

Joint Dynamic Pricing of Multiple Perishable Products Under Consumer Choice

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In response to competitive pressures, firms are increasingly adopting revenue management opportunities afforded by advances in information and communication technologies. Motivated by applications in industry, we consider a dynamic pricing problem facing a firm that sells given initial inventories of multiple substitutable and perishable products over a finite selling horizon. In these applications, since individual product demands are linked through consumer choice processes, the seller must formulate a joint dynamic pricing strategy while explicitly incorporating consumer behavior. For a general model of consumer choice, we model this *multi-product dynamic pricing* problem as a stochastic dynamic program and characterize its optimal prices. In addition, since consumer behavior depends on the nature of product differentiation, we specialize the general choice model to capture vertical and horizontal differentiation. When products are vertically differentiated, our results show monotonicity properties of the optimal prices and reveal that the optimal prices can be determined by considering aggregate inventories of products rather than their individual inventory levels. Accordingly, we develop a polynomial-time exact algorithm for determining the optimal prices. When products are horizontally differentiated, we find that analogous structural properties do not hold and the behavior of optimal prices is substantially different. To solve this problem, we develop a variant of the backward induction algorithm that uses cubic spline interpolation to construct the value functions at each stage. We demonstrate that this interpolation-based algorithm has low memory requirements and generates near-optimal solutions that result in an average error of less than 0.15%.

Key words: revenue management, multiple perishable products, consumer choice, vertical and horizontal differentiation, dynamic programming, structural properties, algorithms

1. Introduction

Increasing global competition is forcing companies to rethink their existing pricing and sales strategies and explore new opportunities afforded by advances in information and communication technologies. With these technologies, firms have extensive reach to customers and to information

systems throughout their organizations, allowing them to collect market data, learn about customer behavior, understand market segments, and change prices dynamically. As a result, sellers can now continuously monitor availability and demand, and adjust prices appropriately to maximize profits. Such demand management principles offer firms an opportunity to make substantial gains – a recent McKinsey study (Marn et al. 2003) estimates that for a typical S&P 1500 company, a 1% improvement in pricing can lead to an 8% improvement in profits. Recognizing these opportunities, managers in several industries including travel, hospitality, public utility, and equipment rental (see Talluri and van Ryzin (2004b) for a detailed profile of revenue management applications in these and other industries) are increasingly using revenue management tools such as dynamic pricing to maximize profits when facing uncertain demand with fixed supply.

In many industrial applications, including those in the travel and hospitality industries, firms offer their customers a line of differentiated products. Since their products are typically perishable (e.g., room inventories for a particular day and seat inventories on a flight), dynamic pricing enables firms in such industries to manage demand effectively and maximize revenues. Formulating a dynamic pricing strategy in these contexts requires managers to determine a *joint* pricing strategy for multiple products since product demands are linked by consumer choice processes. For instance, a hotel might have various types of rooms (e.g., standard vs. deluxe rooms) that differ in the amenities and facilities available for the guest. In this case, the demand for an individual room type depends not only on the price and non-price characteristics of that room type, but also on that of the other room types. As a result, the hotel must understand the choices that consumers make when facing such product assortments and determine the prices for different room-types jointly. Similarly, a discount airline that offers parallel flights (with different arrival and departure times within the same day for a particular origin-destination flight leg) must determine the fares for these itineraries jointly and model how consumers choose these flight legs. While both examples require a joint dynamic pricing strategy, consumer behavior in each context is different and follows the nature of product differentiation. In the hotel example, since the room-types can be ordered based on their quality, they are *vertically* differentiated; consequently, if all the room-types are priced the same, consumers would prefer deluxe rooms over standard rooms. In contrast, in the

airline example, customer preferences are not uniformly ordered and the different flight legs are *horizontally* differentiated. That is, if all flight legs are priced the same, some consumers may prefer an early departure while others may choose a later departure.

Motivated by these revenue management applications, this paper studies the dynamic pricing problem facing a firm that sells given initial inventories of multiple substitutable and perishable products over a finite selling season. The demand for each product depends on the price and non-price characteristics of all the products and the behavior of consumers induced by the nature of product differentiation. We refer to this problem as the *Multi-Product Dynamic Pricing* (MPDP) problem. We formulate a stochastic dynamic program (DP) that embeds a general consumer choice model, which we can specialize for both *vertically* and *horizontally* differentiated products. Our analysis examines the structure of the optimal policy, offers valuable managerial insights, and leads to effective solution methods that can help managers solve practical-sized problems. To the best of our knowledge, this is the first paper that systematically studies the structural properties of joint pricing policies for the MPDP model under consumer choice. We synthesize these new results to develop optimal and near-optimal solution algorithms that have nominal computational requirements.

Academic interest in the area of revenue management has grown with industry adoption; the book by Talluri and van Ryzin (2004b) provides a detailed survey of this literature. We discuss the literature briefly next and defer a more detailed review to Section 2. Papers in this area address demand management decisions for both single and multiple products. The multi-product dynamic pricing literature is more relevant to the MPDP problem. Many of these multi-product dynamic pricing papers incorporate only price factors and do not model consumer choices based on non-price factors. A few recent papers have incorporated consumer choice models, but have focused largely on capacity controls for airline revenue management. In contrast, the MPDP problem we study is broadly applicable, focuses on pricing controls, and recognizes the importance of incorporating the nature of product differentiation by embedding appropriate choice models. In addition, our paper uncovers important analytical insights and develops effective algorithms. We discuss the managerial implications of our results in Section 8.

The MPDP model we develop has applications in several industries. For example, the MPDP model for vertically differentiated products applies to the hospitality (e.g., pricing hotel rooms that offer different amenities), entertainment (e.g., pricing of event tickets for different seat locations), agriculture (e.g., pricing perishable agriculture goods of different grades), and information technology (e.g., pricing of advertisement slots at different positions on web pages) industries. In addition, our analysis of the MPDP model leads to several important contributions. First, for the MPDP problem with a general model of consumer choice, which we refer to as MPDP-G, we characterize the structure of the optimal dynamic prices. This characterization simplifies the solution procedure and is the basis for the structural results for two specialized models — MPDP-V and MPDP-H, representing the MPDP models for vertically and horizontally differentiated products, respectively. Next, for MPDP-V, we provide a complete analysis of the structure of optimal prices. We show that the optimal prices exhibit (1) *quality monotonicity*: the optimal price of a high quality product is always higher than that of a lower quality product; (2) *inventory monotonicity*: when a product's inventory level increases, the firm must set a lower price not only for that product, but also for the other products in the assortment; and (3) *time monotonicity*: as the end of the sales horizon approaches, the firm must reduce the prices for all its products. Further, we show that the price of a product depends on the inventories of the higher quality products only through the aggregate inventory level. In addition, when the product inventories are in surplus (that is, when the total inventory of all products is greater than the maximum possible future demand), we prove that the surplus units, starting from the lowest quality, are of no value to the firm and hence can be removed from inventory. This implies that the pricing policy of the firm should be mainly driven by the respective quality ratings of the products rather than their individual inventory levels. We exploit these structural properties to develop a polynomial-time and exact algorithm that decomposes the multi-dimensional MPDP-V problem into a series of one dimensional DPs based on aggregate inventory levels.

In the MPDP-H model, we illustrate, with a counterexample, that the optimal prices do not conform to the quality, inventory, and time monotonicity properties described earlier. We show that the optimal prices depend on individual inventory availabilities of products, instead of the

aggregate inventory level as in the MPDP-V model. We also prove that for all the products with inventory surplus (that is, the inventory of each product by itself can meet the maximum possible future demand) the firm should charge a uniform price, and for any product with an inventory shortfall (that is, the inventory of each product by itself cannot meet the maximum possible future demand) the firm should charge a premium in addition to the uniform price set for the products with surplus inventories. Consequently, unlike the MPDP-V model, the firm extracts greater value from a product with an inventory shortfall than a product with an inventory surplus, regardless of their respective quality ratings. To solve the MPDP-H effectively, we adopt an approximate dynamic programming approach. This approach is a variant of the backward induction algorithm and stores only a subset of optimal values from previous periods and constructs values for others using a cubic spline function. Our computational tests demonstrate that this algorithm significantly reduces the memory requirement of the backward induction procedure, while still generating near-optimal solutions for practical-sized instances.

The remainder of this paper is organized as follows. Section 2 provides a review of related literature. Section 3 presents a DP formulation of MPDP-G. In Section 4, we derive the pricing structure for MPDP-G and discuss its computational implications. Section 5 studies the analytical properties and presents a solution algorithm for MPDP-V. Section 6 presents the pricing structure of MPDP-H and proposes an interpolation-based algorithm for its solution. Section 7 uses numerical examples to highlight contrasts between MPDP-V and MPDP-H and also reports the effectiveness of the interpolation-based algorithm. Finally, Section 8 describes managerial insights and offers concluding remarks.

2. Related Literature

As the importance of revenue management has grown in practice, so has academic research on dynamic pricing and capacity management. The papers by McGill and van Ryzin (1999), Elmaghraby and Keskinocak (2003), Bitran and Caldentey (2003), and the recent book by Talluri and van Ryzin (2004b) provide comprehensive surveys of the literature in revenue management. We focus our review on models that consider dynamic pricing over a finite selling horizon with no

replenishment opportunities and classify this stream of research into two categories – single and multi-product dynamic pricing models.

The single product dynamic pricing problem entails determining revenue-maximizing prices for a firm that plans to sell a given inventory of a single product over a finite selling horizon. We briefly describe some of the papers in this area. Gallego and van Ryzin (1994) and Zhao and Zheng (2000) present continuous-time formulations of this problem. They derive optimality conditions that show that the optimal price decreases with increasing inventory and when there are fewer time periods remaining. Focusing on contexts where continuous updating of prices might not be feasible, Feng and Gallego (1995), Bitran and Mondschein (1997), and Feng and Xiao (2000) consider models that allow for only a finite number of price changes.

Other researchers have developed multi-product pricing models. In these studies, the pricing decisions for multiple products are linked because of joint capacity constraints and/or due to demand correlations. Given starting inventories of components that a firm uses to build products, Gallego and van Ryzin (1997) model the problem of determining the price for multiple products over a finite selling horizon. Since their model is difficult to solve, they develop heuristics based on the deterministic solution to the problem and show that these are asymptotically optimal. Karaesmen and van Ryzin (2004) consider the substitutability of inventories to determine overbooking limits in a two-period model. When a firm uses a single resource to produce multiple products, Maglaras and Meissner (2006) explore the relation between dynamic pricing and capacity control and show that the dynamic pricing problem of Gallego and van Ryzin (1997) and the capacity control approach (for example, Lee and Hersh 1993) can be reduced to a common formulation. Liu and Milner (2006) study the multi-product pricing problem with a common price constraint. However, these papers do not address consumer choice issues that arise from non-price factors. Talluri and van Ryzin (2004a), and Zhang and Cooper (2005) model consumer choice behavior explicitly when considering booking limit (*capacity control*) policies for airline revenue management. Focusing on a single-leg yield management problem with exogenous fares, Talluri and van Ryzin (2004a) model how consumers choose from multiple fare products in determining the booking limits for various fare classes. Zhang and Cooper (2005) extend their model and consider capacity control

for parallel flights. For an airline revenue management problem that considers parallel flights and consumer choice, Zhang and Cooper (2007) model a *pricing control* problem. They acknowledge the complexity of the DP formulation, construct heuristics based on the pooling of prices and of inventories, and test the performance of the heuristics using a numerical study. We also focus on a pricing control problem; however, unlike previous approaches, in addition to analyzing the MPDP-G problem, we also formulate the MPDP-V and MPDP-H problems, derive strong analytical results, and develop efficient algorithms for specialized models. Our analysis reveals strikingly different pricing characteristics of MPDP-V and MPDP-H, underscoring the importance of studying the MPDP problem with specialized consumer choice models.

3. Model Description and Formulation

Consider the tactical pricing problem facing a firm that sells n substitutable products, indexed as $j = 1, 2, \dots, n$, over a finite selling season. The firm starts the selling season with an initial inventory κ_j of product j and is unable to replenish these inventories during the season. Moreover, the inventories are perishable and any inventory that remains at the end of the season expires. To conveniently model the multi-product demand process, we divide the selling season into T time periods such that each period has at most one customer arrival, and assume that each arriving customer requires no more than one unit of inventory. Let λ_t denote the probability of a customer arrival in period t . We index the time periods in reverse chronological order: that is, $t=0$ and $t=T$ correspond to the *end* and the *beginning* of the selling season. This demand arrival model is similar to others in the revenue management literature (for instance, Gerchak et al. 1985, Talluri and van Ryzin 2004a, and Zhang and Cooper 2007). In each period t for a given current inventory level $\mathbf{x} = (x_1, x_2, \dots, x_n)$, the firm must determine a price vector $\mathbf{p}_t = (p_{1t}, p_{2t}, \dots, p_{nt})$, to maximize the total expected revenue over the T periods. Given price vector \mathbf{p}_t , a consumer chooses to buy a product j with probability $\alpha_j(\mathbf{p}_t)$. The customer may also choose to not purchase a product from the firm — $\alpha_0(\mathbf{p}_t)$ denotes this probability. At inventory level \mathbf{x} , since some products may have zero inventory, we define a state-dependent action space, $\mathcal{P}_{\mathbf{x}}$, in period t , as $\mathcal{P}_{\mathbf{x}} = \{\mathbf{p} \geq \mathbf{0} : \alpha_j(\mathbf{p}) = 0 \text{ if } x_j = 0, j = 1, \dots, n\}$. For a given inventory vector \mathbf{x} , let $V_t(\mathbf{x})$ denote the optimal expected

revenue from period t to the end of the season. Then the MPDP-G problem can be formulated as the following dynamic program:

$$V_t(\mathbf{x}) = \max_{\mathbf{p}_t \in \mathcal{P}_\mathbf{x}} \left\{ \sum_{j=1}^n \lambda_t \alpha_j(\mathbf{p}_t) (p_{jt} + V_{t-1}(\mathbf{x} - \mathbf{e}_j)) + \lambda_t \alpha_0(\mathbf{p}_t) V_{t-1}(\mathbf{x}) + (1 - \lambda_t) V_{t-1}(\mathbf{x}) \right\}, \quad (1)$$

with boundary conditions $V_t(\mathbf{0}) = 0$ for $t = 0, 1, \dots, T$, and $V_0(\mathbf{x}) = 0$ for all \mathbf{x} , where \mathbf{e}_j is a vector of size n with 1 at the j^{th} entry and zeros elsewhere. Since $\alpha_0(\mathbf{p}_t) = 1 - \sum_{j=1}^n \alpha_j(\mathbf{p}_t)$, we can rewrite the optimality equation (1) as:

$$V_t(\mathbf{x}) = \max_{\mathbf{p}_t \in \mathcal{P}_\mathbf{x}} \left\{ \sum_{j=1}^n \lambda_t \alpha_j(\mathbf{p}_t) (p_{jt} + V_{t-1}(\mathbf{x} - \mathbf{e}_j) - V_{t-1}(\mathbf{x})) \right\} + V_{t-1}(\mathbf{x}). \quad (2)$$

To further clarify the firm's dynamic pricing problem in (2), we define the difference functions of $V_t(\mathbf{x})$ with respect to t and x_j . Let

$$\Delta_t V_t(\mathbf{x}) = V_t(\mathbf{x}) - V_{t-1}(\mathbf{x}) \quad \text{for } t=0,1,\dots,T, \text{ and,}$$

$$\Delta_{x_j} V_t(\mathbf{x}) = V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_j) \quad \text{for } j = 1, 2, \dots, n.$$

Note that $\Delta_t V_t(\mathbf{x})$ represents the maximum expected gain, in period t at inventory level \mathbf{x} , if the firm had one additional selling period (marginal value of *time*). Similarly, $\Delta_{x_j} V_t(\mathbf{x})$ is the maximum expected gain, in period t at inventory levels \mathbf{x} , if the firm had one more unit of product j inventory to sell (marginal value of *inventory*). Using this notation, we rewrite (2) as:

$$\Delta_t V_t(\mathbf{x}) = V_t(\mathbf{x}) - V_{t-1}(\mathbf{x}) = \max_{\mathbf{p}_t \in \mathcal{P}_\mathbf{x}} \left\{ \sum_{j=1}^n \lambda_t \alpha_j(\mathbf{p}_t) (p_{jt} - \Delta_{x_j} V_{t-1}(\mathbf{x})) \right\} = \max_{\mathbf{p}_t \in \mathcal{P}_\mathbf{x}} \{G_t(\mathbf{x}, \mathbf{p}_t)\}. \quad (3)$$

Note that in (3), the purchase probabilities $\alpha_j(\mathbf{p}_t)$ do not make assumptions about either product characteristics or the nature of product differentiation. Therefore, we refer to it as the *general* model of consumer choice, and denote the pricing problem as by MPDP-G.

4. Pricing Under a General Model of Consumer Choice

In this section, we study the pricing structure of MPDP-G, which will allow us to develop, in subsequent sections, tailored methodologies for MPDP with vertically and horizontally differentiated products. In Section 4.1, we derive the optimal prices of the MPDP-G. Our analysis reveals an intuitive expression for the optimal price of a product, at a given time and a given inventory level,

as the sum of the *current* and *future* values of a unit of inventory of that product. Building on this result, we present the structural properties a *single product* problem in Section 4.2. The properties of this special case are essential for the analysis that appears later.

4.1. Optimal Multi-Product Prices

Recall from (3) that $G_t(\mathbf{x}, \mathbf{p}_t)$ is the expected extra gain realized in period t by selling a single unit of inventory \mathbf{x} (of any product) when prices are set at \mathbf{p}_t . Specifically,

$$G_t(\mathbf{x}, \mathbf{p}_t) = \sum_{j=1}^n \lambda_t \alpha_j(\mathbf{p}_t) (p_{jt} - \Delta_{x_j} V_{t-1}(\mathbf{x})), \quad \mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}. \quad (4)$$

We can interpret $G_t(\mathbf{x}, \mathbf{p}_t)$ as the marginal value of *time* at inventory level \mathbf{x} in period t when prices are set at \mathbf{p}_t . To proceed, we need the following additional notation. Define $\mathbf{h}(\mathbf{p}_t) = (h_1(\mathbf{p}_t), \dots, h_n(\mathbf{p}_t))$ as an n -dimensional vector that satisfies

$$\mathbf{h}(\mathbf{p}_t) = -\boldsymbol{\alpha}(\mathbf{p}_t) \left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} \right)^{-1}, \quad (5)$$

where $\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t}$ is the Jacobian matrix of the choice probability vector $\boldsymbol{\alpha}(\mathbf{p}_t) = (\alpha_1(\mathbf{p}_t), \alpha_2(\mathbf{p}_t), \dots, \alpha_n(\mathbf{p}_t))$ when prices are set at \mathbf{p}_t :

$$\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} = \begin{bmatrix} \frac{\partial \alpha_1(\mathbf{p}_t)}{\partial p_{1t}} & \dots & \frac{\partial \alpha_1(\mathbf{p}_t)}{\partial p_{nt}} \\ \vdots & \ddots & \vdots \\ \frac{\partial \alpha_n(\mathbf{p}_t)}{\partial p_{1t}} & \dots & \frac{\partial \alpha_n(\mathbf{p}_t)}{\partial p_{nt}} \end{bmatrix}. \quad (6)$$

The following theorem characterizes the optimal prices.

Theorem 1 *For given \mathbf{x} and t , suppose $G_t(\mathbf{x}, \mathbf{p}_t)$ is strictly quasi-concave in \mathbf{p}_t . Then, the optimal price vector, denoted by $\mathbf{p}_t(\mathbf{x}) = (p_{1t}(\mathbf{x}), \dots, p_{nt}(\mathbf{x}))$, satisfies:*

$$\mathbf{p}_t(\mathbf{x}) = \mathbf{h}(\mathbf{p}_t(\mathbf{x})) + \Delta_{\mathbf{x}} V_{t-1}(\mathbf{x}), \quad (7)$$

where $\Delta_{\mathbf{x}} V_t(\mathbf{x}) = (\Delta_{x_1} V_t(\mathbf{x}), \dots, \Delta_{x_n} V_t(\mathbf{x}))$. Specifically, the j^{th} element of $\mathbf{p}_t(\mathbf{x})$ is given by

$$p_{jt}(\mathbf{x}) = h_j(\mathbf{p}_t(\mathbf{x})) + \Delta_{x_j} V_{t-1}(\mathbf{x}), \quad x_j > 0. \quad (8)$$

Proof. See Appendix EC.1.

Equation (8) in Theorem 1 shows that the optimal price of product j at inventory level \mathbf{x} in period t is composed of two terms. The first term, $h_j(\mathbf{p}_t)$, is the *current* marginal value of the

product j 's inventory, and the second term, $\Delta_{x_j} V_{t-1}(\mathbf{x})$, is the *future* marginal value of the product j 's inventory, at inventory level \mathbf{x} .

Theorem 1 is also valuable from an algorithmic standpoint. Without this result, determining the optimal prices for (3) would entail using numerical methods to find the optimal solution at each inventory level and time period. Such numerical methods can be computationally challenging for practical-sized MPDP-G problems, known in the literature as *curse of dimensionality* (Powell 2007). To illustrate this challenge, we consider a three product problem with a starting inventory of 10 units of each product. Then, we have 11^3 possible inventory levels in each period. Suppose a numerical method considers 100 different price values for each product as candidate solutions. Then we have to perform $11^3 \times 100^3$ calculations to compute the value function $V_t(\mathbf{x})$ in each period and repeat this step for all T periods. In contrast, Theorem 1 allows us to *directly compute* (instead of having to solve an optimization problem numerically) the optimal prices using (8) and determine $V_t(\mathbf{x})$.

In the next subsection, we consider a single product model with general consumer choice. The results from this special case are valuable in developing results for the multi-product problems.

4.2. Preliminary Results: Properties of the Optimal Single Product Price

When the firm offers only a single product, consumers must choose between purchasing this product and the outside option. In this case, optimality equation (3) simplifies to

$$\Delta_t V_t(x) = \max_{p_t} \{G_t(x, p_t)\},$$

where

$$G_t(x, p_t) = \lambda_t \alpha(p_t) (p_t - \Delta_x V_{t-1}(x)), x > 0.$$

This single product dynamic pricing problem has been studied by Zhang and Cooper (2007) and Zhao and Zheng (2000) in a continuous-time framework. For this special case of the MPDP-G problem, we can show the results in the following theorem.

Theorem 2 *For any inventory $x > 0$, let $G_t(x, p_t)$ be a strictly quasiconcave function of p_t , and let $p_t(x)$ be the optimal price in period t . Then*

- (a) $\Delta_x V_t(x)$ and $p_t(x)$ are nondecreasing functions of t .
- (b) $\Delta_x V_t(x)$ and $p_t(x)$ are nonincreasing functions of x .
- (c) If $\lambda_t \geq \lambda_{t+1}$, then $\Delta_t V_t(x)$ is a non-increasing function of t

Theorem 2 implies that $V_t(x)$ is a supermodular function of t and x , a concave function of x , and a concave function of t if λ_t is non-increasing in t . Zhang and Cooper (2007) report the inventory monotonicity property and for a continuous-time single product model, Zhao and Zheng (2000) show that both part (a) (*time monotonicity*) and part (b) (*inventory monotonicity*) of Theorem 2 hold under certain conditions. Since Theorems 1 and 2 both require quasiconcavity of $G_t(x, p_t)$, we next identify conditions on the choice probabilities that are sufficient to ensure this property in our discrete-time model.

Lemma 3 *If $\alpha(p_t)$ is decreasing and differentiable in p , and $h(p) = -\frac{\alpha(p)}{\alpha'(p)}$ is a non-increasing function of p , then $G_t(x, p)$ is a strictly quasiconcave function of p for any $t \geq 1$ and $x > 0$.*

Proof. See Appendix EC.2.

5. Pricing of Vertically Differentiated Products

An assortment of products is vertically differentiated when the products in the assortment can be ordered on the basis of their attributes, say product *quality*, from the highest to the lowest. Examples include hotels with multiple types of rooms offering a vertically differentiated assortment to their guests and entertainment venues selling different seat locations to their patrons. In such contexts, product quality impacts the choice of consumers and hence must be explicitly incorporated in the consumer's purchase probabilities. Accordingly, we first develop a quality-based consumer choice model for vertically differentiated products (Section 5.1). We then formulate the MPDP problem with the quality-based choice model, which we refer to as the MPDP-V problem (Section 5.2). We also present the structural properties of the optimal prices and discuss their managerial implications (Section 5.3). Finally, we synthesize the structural results to develop an effective and exact algorithm to solve the MPDP-V problem (Section 5.4).

5.1. A Quality-Based Choice Model for Vertical Differentiation

We adapt a classical approach in Tirole (1988) to model consumer choice in this context. Consider a consumer who must choose from n products with different quality ratings. Let product j have a quality index q_j and assume that product quality can be ordered as $q_1 > q_2 > \dots > q_n$. Suppose that the consumer's sensitivity to quality can be parameterized as a scalar θ that is uniformly distributed between 0 and 1. Then, the consumer's utility when purchasing product j at price p_{jt} is $u_{jt} = \theta q_j - p_{jt}$. In this setup, if any two products i and j have the same price, then all the consumers would choose product j over product i when $q_j > q_i$. Recall that the consumer can also purchase the product elsewhere; for convenience, we normalize the value of this outside option to 0. Therefore, a rational customer will buy product j if $u_j \geq \max_{k \neq j} \{0, u_k\}$.

Let us now examine the set of candidate prices \mathbf{p}_t in period t . Observe that, if the *price-to-quality* ratio, $\frac{p_{jt}}{q_j}$, of product j is lesser than that of product $j+1$, then the customer would have a greater utility when purchasing product j . As a result, we can argue that, if $\frac{p_{jt}}{q_j} \leq \frac{p_{j+1,t}}{q_{j+1}}$,

$$\alpha_{j+1}(p_{1t}, \dots, p_{jt}, p_{j+1,t}, \dots, p_{nt}) = \alpha_{j+1}(p_{1t}, \dots, p_{jt}, q_{j+1} \frac{p_{jt}}{q_j}, \dots, p_{nt}) = 0. \quad (9)$$

Therefore, we need to consider only the prices that satisfy $\frac{p_{jt}}{q_j} \geq \frac{p_{j+1,t}}{q_{j+1}}$, for $j = 1, \dots, n-1$. We can derive a similar restriction based on the *price difference-to-quality difference* ratio of *adjacent* products as follows. Suppose $\frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} \leq \frac{p_{j+1,t} - p_{j+2,t}}{q_{j+1} - q_{j+2}}$ for some $j < n-1$. Then,

$$\begin{aligned} \alpha_{j+1}(\mathbf{p}_t) &= P(\theta q_{j+1} - p_{j+1,t} \geq 0, \theta q_{j+1} - p_{j+1,t} \geq \theta q_k - p_{kt}, k \neq j+1) \\ &= P\left(\begin{array}{l} \theta q_{j+1} - p_{j+1,t} \geq 0 \\ \frac{p_{j+1,t} - p_{j+2,t}}{q_{j+1} - q_{j+2}} \leq \theta \leq \frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} \\ \theta q_{j+1} - p_{j+1,t} \geq \theta q_k - p_{kt}, k \neq j, j+1 \end{array}\right). \end{aligned} \quad (10)$$

Since the second event in (10) is a null event, we must have $\alpha_{j+1}(\mathbf{p}_t) = 0$, that is, a customer would prefer one of the adjacent products, j or $j+2$, to product $j+1$. Further, when $\frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} \leq \frac{p_{j+1,t} - p_{j+2,t}}{q_{j+1} - q_{j+2}}$, the choice probabilities of other products, $\alpha_k(\mathbf{p}_t)$, $k \neq j$, do not depend on $p_{j+1,t}$. Together, these observations imply that, given \mathbf{x} , it is sufficient to restrict the candidate prices to the following set of *quality-aligned prices*, which we denote as $\tilde{\mathcal{P}}_{\mathbf{x}}$.

$$\tilde{\mathcal{P}}_{\mathbf{x}} = \left\{ \mathbf{p}_t \in \mathcal{P}_{\mathbf{x}} : \begin{array}{l} 1 \geq \frac{p_{1t}}{q_1} \geq \frac{p_{2t}}{q_2} \geq \dots \geq \frac{p_{nt}}{q_n} \geq 0 \\ 1 \geq \frac{p_{1t} - p_{2t}}{q_1 - q_2} \geq \frac{p_{2t} - p_{3t}}{q_2 - q_3} \geq \dots \geq \frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n} \geq 0 \end{array} \right\}. \quad (11)$$

Limiting our attention to the prices in set $\tilde{\mathcal{P}}_{\mathbf{x}}$, we derive closed form expressions for the purchase probabilities $\alpha_j(\mathbf{p}_t)$, for $j = 1, 2, \dots, n$, in the following Lemma.

Lemma 4 For any $\mathbf{p}_t \in \tilde{\mathcal{P}}_{\mathbf{x}}$,

- (a) $\frac{p_{jt} - p_{kt}}{q_j - q_k}$ is a decreasing sequence of k for a fixed $k > j$,
- (b) $\frac{p_{kt} - p_{jt}}{q_k - q_j}$ is a decreasing sequence of k for a fixed $k < j$,
- (c) the choice probabilities are given by

$$\alpha_j(\mathbf{p}_t) = \begin{cases} 1 - \frac{p_{1t} - p_{2t}}{q_1 - q_2}, & j = 1, \\ \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j} - \frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}}, & j = 2, \dots, n-1, \\ \frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n} - \frac{p_{nt}}{q_n} & j = n, \text{ and,} \\ \frac{p_{nt}}{q_n} & j = 0. \end{cases} \quad (12)$$

Proof. In Appendix EC.3.

5.2. Optimal Prices for Vertically Differentiated Products

After substituting the purchase probabilities from Lemma 4(c) into $G_t(\mathbf{x}, \mathbf{p}_t)$ in (4), we obtain:

$$\begin{aligned} G_t(\mathbf{x}, \mathbf{p}_t) &= \lambda_t \left(1 - \frac{p_{1t} - p_{2t}}{q_1 - q_2} \right) (p_{1t} - \Delta_{x_1} V_{t-1}(\mathbf{x})) \\ &+ \sum_{k=2}^{n-1} \lambda_t \left(\frac{p_{k-1,t} - p_{kt}}{q_{k-1} - q_k} - \frac{p_{kt} - p_{k+1,t}}{q_k - q_{k+1}} \right) (p_{kt} - \Delta_{x_k} V_{t-1}(\mathbf{x})) \\ &+ \lambda_t \left(\frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n} - \frac{p_{nt}}{q_n} \right) (p_{nt} - \Delta_{x_n} V_{t-1}(\mathbf{x})), \quad \mathbf{p}_t \in \tilde{\mathcal{P}}_{\mathbf{x}}. \end{aligned} \quad (13)$$

Substituting (13) in (3) gives the optimality equation for the MPDP-V problem. Clearly, $G_t(\mathbf{x}, \mathbf{p}_t)$ in (13) is a quadratic and concave function of \mathbf{p}_t ; consequently, Theorem 1 applies to MPDP-V. Next, we use (5)-(7) to determine the optimal prices, $\mathbf{p}_t(\mathbf{x})$, for this problem. The Jacobian matrix of the purchase probability vector $\boldsymbol{\alpha}(\mathbf{p}_t)$ for given $\mathbf{p}_t \in \tilde{\mathcal{P}}_{\mathbf{x}}$ works out to be

$$\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} = \begin{pmatrix} \frac{-1}{q_1 - q_2} & \frac{1}{q_1 - q_2} & 0 & 0 & \dots & 0 \\ \frac{1}{q_1 - q_2} & \frac{-(q_1 - q_3)}{(q_1 - q_2)(q_2 - q_3)} & \frac{1}{q_2 - q_3} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{q_{n-1} - q_n} & \frac{-q_{n-1}}{(q_{n-1} - q_n)q_n} \end{pmatrix}. \quad (14)$$

The Jacobian of $\boldsymbol{\alpha}(\mathbf{p}_t)$ in (14) is a triangular symmetric matrix with negative diagonal entries. This means that when the price of product j ($1 < j < n$) increases, the likelihood that a customer will buy that product decreases at a *linear* rate and the probability that the customer will buy one of its adjacent products increases at a *linear* rate. Further, since $\frac{\partial \alpha_k(\mathbf{p}_t)}{\partial p_{jt}} = 0$, for $k \neq j-1, j, j+1$,

an infinitesimal change in the price of product j will not affect the likelihood that a customer will choose product k , $k \neq j-1, j, j+1$. Also, $\frac{\partial \alpha_0(\mathbf{p}_t)}{\partial p_{nt}} = \frac{1}{q_n}$ and $\frac{\partial \alpha_0(\mathbf{p}_t)}{\partial p_{jt}} = 0$ for $j \neq 1, n$, implying that the no-purchase probability depends *only* on p_{nt} , the price of the lowest quality product.

We can derive the inverse of the Jacobian matrix in (14) as:

$$\left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t}\right)^{-1} = - \begin{pmatrix} q_1 & q_2 & \cdots & q_{n-1} & q_n \\ q_2 & q_2 & \cdots & q_{n-1} & q_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ q_{n-1} & q_{n-1} & \cdots & q_{n-1} & q_n \\ q_n & q_n & \cdots & q_n & q_n \end{pmatrix}. \quad (15)$$

Substituting (14) and (15) into $\mathbf{h}(\mathbf{p}_t) = -\boldsymbol{\alpha}(\mathbf{p}_t) \left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t}\right)^{-1}$, we get:

$$\begin{aligned} h_j(\mathbf{p}_t) &= \left(1 - \frac{p_{1t} - p_{2t}}{q_1 - q_2}, \dots, \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j} - \frac{p_{j+1,t} - p_{jt}}{q_{j+1} - q_j}, \dots, \frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n} - \frac{p_{nt}}{q_n}\right) \times \begin{pmatrix} q_1 \\ \vdots \\ q_j \\ \vdots \\ q_n \end{pmatrix} \\ &= q_j - p_{jt}, \quad j = 1, 2, \dots, n. \end{aligned} \quad (16)$$

Since the optimal prices satisfy $p_{jt}(\mathbf{x}) = q_j - p_{jt}(x) + \Delta_{x_j} V_{t-1}(\mathbf{x})$ due to Theorem 1, we obtain

$$p_{jt}(\mathbf{x}) = \frac{1}{2} (q_j + \Delta_{x_j} V_{t-1}(\mathbf{x})), \quad j = 1, 2, \dots, n. \quad (17)$$

As seen, the optimal price $p_{jt}(\mathbf{x})$ is the average of the quality rating of product j , q_j , and the future marginal value of product j inventory, $\Delta_{x_j} V_{t-1}(\mathbf{x})$. Using (13), the optimal value function becomes

$$\begin{aligned} V_t(\mathbf{x}) &= G_t(\mathbf{x}, \mathbf{p}_t(\mathbf{x})) + V_{t-1}(\mathbf{x}) \\ &= \lambda_t \left(q_1 - 2p_{1t}(\mathbf{x}) + \sum_{k=1}^{n-1} \frac{(p_{kt}(\mathbf{x}) - p_{k+1,t}(\mathbf{x}))^2}{q_k - q_{k+1}} + \frac{p_{nt}^2(\mathbf{x})}{q_n} \right) + V_{t-1}(\mathbf{x}). \end{aligned} \quad (18)$$

Equations (17) and (18) allow us to derive structural properties of the optimal price $\mathbf{p}_t(\mathbf{x})$ and the optimal value function $V_t(\mathbf{x})$. We describe these properties next.

5.3. Structural Properties of MPDP-V

In the following discussion, we assume, without loss of generality, that each element of \mathbf{x} is positive. Recall that $p_{jt}(\mathbf{x})$ is the optimal price of product j in period t given inventory state \mathbf{x} . Denote $|\mathbf{x}|_j = \sum_{i=1}^j x_i$ as the aggregate inventory of the first j products given \mathbf{x} . The next theorem examines the behavior of the optimal prices as a function of the aggregate inventory $|\mathbf{x}|_j$.

Theorem 5 *The optimal price $\mathbf{p}_t(\mathbf{x})$ and value function $V_t(\mathbf{x})$ have the following properties:*

- (a) *If $|\mathbf{x}|_j \geq t$, then $p_{kt}(\mathbf{x}) = \frac{q_k}{2}$ for $k \geq j$ and $\alpha_k(\mathbf{p}_t(\mathbf{x})) = 0$ for $k > j$.*
- (b) *If $|\mathbf{x}|_j \geq t$, then $p_{it}(\mathbf{x}) = p_{it}(\mathbf{x} + \mathbf{e}_k)$ for $i \leq j \leq k$ and $V_t(\mathbf{x}) = V_t(\mathbf{x} + \mathbf{e}_k)$ for $k \geq j$.*
- (c) *$p_{jt}(\mathbf{x})$ depends on the inventory levels of the first j products only through their sum $|\mathbf{x}|_j$, i.e., $p_{jt}(\mathbf{x}) = p_{jt}(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'})$ for $i < i' \leq j$.*
- (d) *The price difference of two adjacent products, $p_{j-1,t}(\mathbf{x}) - p_{jt}(\mathbf{x})$, is nonnegative, and depends only on the total inventory of the first $j - 1$ products $|\mathbf{x}|_{j-1}$.*

Proof. See Appendix EC.4.

Theorem 5 has several interesting managerial and operational implications. First, when the firm has enough inventory of the high quality products to satisfy the potential remaining demand (i.e., $|\mathbf{x}|_j \geq t$ for some $j < n$), the firm should set the price for each of the remaining lower quality products at a constant level ($p_{it}(\mathbf{x}) = \frac{q_i}{2}$, $i > j$), for the rest of the selling season. In this case, since the price of a low quality product satisfies the price-to-quality ratio $\frac{p_{it}(\mathbf{x})}{q_i} = \frac{p_{i+1,t}(\mathbf{x})}{q_{i+1}}$ for $i \geq j$, the purchase probability for the low quality product is zero (see arguments in Section 5.1). Consequently, the firm should focus on selling the high quality products alone for the remaining periods. Part (b) states that, when $|\mathbf{x}|_j \geq t$, the optimal prices of the first j products are not affected by the inventory increments of the last $n - j$ products. As we will see in Section 6, this is not true when the products are horizontally differentiated. Part (c) states that, when setting the price $p_{jt}(\mathbf{x})$ for product j , the firm should treat the inventories (x_1, \dots, x_j) of the first j high quality products at the *aggregate level* $|\mathbf{x}|_j$. Note that $p_{jt}(\mathbf{x})$ still depends on the inventories (x_{j+1}, \dots, x_n) of the last $n - j$ low quality products at their *individual levels*. However, as stated in part (d), the *price difference* of two adjacent products, $p_{j-1,t}(\mathbf{x}) - p_{jt}(\mathbf{x})$, is nonnegative and depends only on the aggregate inventory $|\mathbf{x}|_{j-1}$ and is *independent* of all else. Equivalently, if we view price $p_{j-1,t}(\mathbf{x})$ as a (nonnegative) *markup* of price $p_{j,t}(\mathbf{x})$, then part (d) says that only the aggregate inventory $|\mathbf{x}|_{j-1}$ is relevant in determining this markup. The nonnegative markup also means that in MPDP-V, a high quality product is never priced less than a low quality product, regardless of their individual inventory levels. We exploit these properties to develop a *polynomial-time* and *exact* algorithm in Section 5.4.

Theorem 6 *For the MPDP-V model,*

- (a) $\Delta_{x_j} V_t(\mathbf{x})$ is non-decreasing in t (or equivalently, $\Delta_t V_t(\mathbf{x})$ is non-decreasing in x_j).
- (b) $\Delta_{x_j} V_t(\mathbf{x})$ is non-increasing in x_i , $i \neq j$, (or equivalently, $\Delta_{x_i} V_t(\mathbf{x})$ is non-increasing in x_j).
- (c) $\Delta_{x_j} V_t(\mathbf{x})$ is non-increasing in x_j .
- (d) If $\lambda_t \geq \lambda_{t+1}$, then $\Delta_t V_t(\mathbf{x})$ is non-increasing in t .

Proof. See Appendix EC.5.

Theorem 6 implies that $V_t(\mathbf{x})$ is supermodular in time t and product j inventory x_j , submodular in inventory levels of products i and j (x_i and x_j respectively), and concave in product j inventory x_j . Further, $V_t(\mathbf{x})$ is also concave in time t if the arrival probability λ_t is non-increasing in t . Since the optimal price satisfies $p_{jt}(\mathbf{x}) = \frac{1}{2} (q_j + \Delta_{x_j} V_{t-1}(\mathbf{x}))$, we know that $p_{jt}(\mathbf{x})$ carries all the structural properties of $\Delta_{x_j} V_{t-1}(\mathbf{x})$. The next corollary is a direct consequence of Theorem 6.

Corollary 7 *The optimal price $p_{jt}(\mathbf{x})$ is a non-decreasing function of t , a non-increasing function of j , a non-increasing function of x_j , and a non-increasing function of x_i for $i \neq j$.*

Corollary 7 generalizes the monotonicity results for the single product case in Section 4.2 to MPDP-V. Specifically, Corollary 7 shows that the optimal prices exhibit (1) quality monotonicity: a high quality product is always priced no less than a low quality product; (2) inventory monotonicity: $p_{jt}(\mathbf{x})$ becomes lower if the inventory of *any product* becomes higher; and (3) time monotonicity: the price for any product is non-increasing when the end of the sales horizon approaches. As we will see in Section 7, these monotonicity properties do not necessarily hold for the horizontally differentiated products.

5.4. An Efficient and Exact Algorithm for MPDP-V

In this section, we demonstrate that the structural results in Section 5.3 can be effectively translated into a computational algorithm that is capable of solving practical-sized problems. Exploiting the structure of the optimal prices and values, the algorithm decomposes the multi-dimensional state and action space of MPDP-V into a sequence of single-dimensional state and action space DPs, thereby drastically reducing the computational effort, both in memory requirements and running time. The algorithm starts with the lowest quality product and iteratively computes, in polynomial

time, $p_{jt}(\mathbf{x})$ and $V_t(\mathbf{x})$ using the optimal values of the single-dimensional DPs.

To describe this algorithm, we first define the following notation that helps us identify those products that have positive inventories at a given inventory level. For any inventory level \mathbf{x} , suppose that \mathbf{x} has $m(\mathbf{x})$ ($m(\mathbf{x}) \leq n$) products with positive inventories. Let $k_r(\mathbf{x})$, $r = 1, \dots, m(\mathbf{x})$, be the product with the r^{th} highest quality rating among the products with non-zero inventory levels. Accordingly, $\{k_1(\mathbf{x}), \dots, k_{m(\mathbf{x})}(\mathbf{x})\}$ represents the set of products with non-zero inventories at inventory level \mathbf{x} . For notational convenience, hereafter, we express $k_r(\mathbf{x})$ simply as k_r , and $m(\mathbf{x})$ as m . Based on Theorem 5(c), the price of a product k_r depends on the inventories of the products with equal or better qualities only through aggregate inventory $|\mathbf{x}|_{k_r}$. Following this principle, for every product k_r , $r = 1, \dots, m$, our algorithm first solves a single-dimensional DP that determines the optimal expected value in period t with $0 \leq x = |\mathbf{x}|_{k_r} \leq |\boldsymbol{\kappa}|_{k_r}$ units of inventory of product k_r . Let the value functions of this DP be $V_t^{k_r}(x)$. For inventory level \mathbf{x} , we start with the lowest quality product k_m with a positive inventory, and directly obtain its price $p_{k_m,t}(\mathbf{x}) = p_{k_m,t}(|\mathbf{x}|_{k_m})$ from $V_t^{k_m}(|\mathbf{x}|_{k_m})$. Next, the algorithm progresses iteratively from k_m to k_1 and at each stage obtains prices $p_{k_{r-1},t}(\mathbf{x})$ using previously computed prices $p_{k_r,t}(\mathbf{x})$ by expressing the price difference of two *adjacent* products with positive inventories, from (17), as follows:

$$p_{k_{r-1},t}(\mathbf{x}) - p_{k_r,t}(\mathbf{x}) = \frac{1}{2} (q_{k_{r-1}} - q_{k_r} + V_{t-1}(\mathbf{x} - \mathbf{e}_{k_r}) - V_{t-1}(\mathbf{x} - \mathbf{e}_{k_{r-1}})).$$

Recall that Theorem 5(d) states that the above price difference depends only on $|\mathbf{x}|_{k_{r-1}}$. Therefore, when computing this difference, we may treat state $\mathbf{x} - \mathbf{e}_{k_r}$ as if it were $(0, \dots, 0, |\mathbf{x}|_{k_{r-1}}, 0, \dots, 0)$ and treat state $\mathbf{x} - \mathbf{e}_{k_{r-1}}$ as if it were $(0, \dots, 0, |\mathbf{x}|_{k_{r-1}} - 1, 0, \dots, 0, 1, 0, \dots, 0)$, where value 1 is in the k_r^{th} position of this vector. Now, define $V_t^{k_{r-1}, k_r}(x-1, 1)$ as the optimal expected value in period t when we have $x-1$ units of product k_{r-1} , a single unit of product k_r , and zero inventory of all other products. Consequently,

$$p_{k_{r-1},t}(\mathbf{x}) - p_{k_r,t}(\mathbf{x}) = \frac{1}{2} \left(q_{k_{r-1}} - q_{k_r} + V_{t-1}^{k_{r-1}}(|\mathbf{x}|_{k_{r-1}}) - V_{t-1}^{k_{r-1}, k_r}(|\mathbf{x}|_{k_{r-1}} - 1, 1) \right). \quad (19)$$

Therefore, in general, to determine $p_{jt}(\mathbf{x})$ for any \mathbf{x} , it is sufficient to know, for all $x \leq \min\{t, |\boldsymbol{\kappa}|_j\}$, value functions $V_{t-1}^j(x)$ for $j = 1, \dots, n$, and value functions $V_{t-1}^{jh}(x-1, 1)$ for all $h > j$, and $j =$

$1, \dots, n-1$. Finally, the algorithm generates the optimal values of $V_t(\mathbf{x})$ by combining the values of relevant functions V_t^j and V_t^{jh} .

Algorithm 8 (An Exact Algorithm for MPDP-V)

Step 1 Determine the value functions $V_t^j(x)$ and $V_t^{jh}(x, 1)$:

1a For all t and $1 \leq x \leq \min\{|\boldsymbol{\kappa}|_j, t\}$, compute $V_t^j(x)$ for $j = 1, \dots, n$, and $V_t^{jh}(x, 1)$ for all $h > j$ and $j = 1, \dots, n-1$.

1b For all t and $t \leq x \leq |\boldsymbol{\kappa}|_j$, set

$$\begin{aligned} V_t^j(x) &= V_t^j(t) \text{ for } j = 1, \dots, n \text{ and,} \\ V_t^{jh}(x, 1) &= V_t^j(t) \text{ for } h > j, \quad j = 1, \dots, n-1. \end{aligned} \quad (20)$$

Step 2 Determine the optimal prices $p_{k_r, t}(\mathbf{x})$, $r = 1, \dots, m$:

2a For all t and $\mathbf{x} \leq \boldsymbol{\kappa}$, set the optimal price $p_{k_m, t}(\mathbf{x})$ as

$$p_{k_m, t}(\mathbf{x}) = \begin{cases} \frac{q_{k_m}}{2} & \text{if } t \leq |\mathbf{x}|_{k_m}, \\ \frac{1}{2} (q_{k_m} + V_{t-1}^{k_m}(|\mathbf{x}|_{k_m}) - V_{t-1}^{k_m}(|\mathbf{x}|_{k_m} - 1)) & \text{if } t > |\mathbf{x}|_{k_m}. \end{cases} \quad (21)$$

2b For all t and $\mathbf{x} \leq \boldsymbol{\kappa}$, starting with $r = m$ and $2 \leq r \leq m$, set $p_{k_{r-1}, t}(\mathbf{x})$, as

$$p_{k_{r-1}, t}(\mathbf{x}) = \begin{cases} \frac{q_{k_{r-1}}}{2} & \text{if } t \leq |\mathbf{x}|_{k_{r-1}}, \\ p_{k_r, t}(\mathbf{x}) + \frac{1}{2} \begin{pmatrix} q_{k_{r-1}} - q_{k_r} + V_{t-1}^{k_{r-1}}(|\mathbf{x}|_{k_{r-1}}) \\ -V_{t-1}^{k_{r-1}, k_r}(|\mathbf{x}|_{k_{r-1}} - 1, 1) \end{pmatrix} & \text{if } t > |\mathbf{x}|_{k_{r-1}}. \end{cases} \quad (22)$$

Step 3 Compute the value function $V_t(\mathbf{x})$:

3a For all t , $1 \leq r < m$, $0 \leq \mathbf{x} \leq \boldsymbol{\kappa}$ and $|\mathbf{x}|_{k_m} \leq t$, set

$$V_t(\mathbf{x}) = \sum_{r=1}^m \sum_{x=1}^{|\mathbf{x}|_{k_r}} V_t^{k_r}(x) - \sum_{r=2}^m \sum_{x=1}^{|\mathbf{x}|_{k_{r-1}}-1} V_t^{k_{r-1}, k_r}(x, 1). \quad (23)$$

3b For all t , $0 \leq \mathbf{x} \leq \boldsymbol{\kappa}$ and $|\mathbf{x}|_{k_m} > t$, let $k_{r'} = \min\{r : |\mathbf{x}|_{k_r} \geq t\}$, and set

$$V_t(\mathbf{x}) = V_t(x_1, \dots, x_{k_{r'}-1}, t - |\mathbf{x}|_{k_{r'}-1}, 0, \dots, 0).$$

A brief interpretation of Algorithm 8 is in order. Step 1a computes the one-dimensional value functions $V_t^j(x)$ and $V_t^{jh}(x, 1)$. Next, Step 1b uses Theorem 5(b) and removes the redundant lower quality inventories in period t , when the aggregate inventory at a certain quality level is greater than

t . Following Theorem 5(a)-(c), which states that $p_{k_m,t}(\mathbf{x}) = p_{k_m,t}(0, \dots, 0, |\mathbf{x}|_{k_m}, 0, \dots, 0)$, equation (21) determines the optimal price for product k_m first. Now, suppose we have iteratively computed $p_{k_r,t}(\mathbf{x})$ and now need to determine $p_{k_{r-1},t}(\mathbf{x})$. Then, if the total inventory of the first $r-1$ products with positive inventories is greater than the remaining periods, $|\mathbf{x}|_{k_{r-1}} > t$, we set the price of product k_{r-1} at constant $\frac{q_{k_{r-1}}}{2}$, resulting in the first expression of (22). Otherwise, we use the second expression of (22) to determine the price of product k_{r-1} , as explained in (19). Finally, Step 3 derives the optimal values directly from the values of the single dimensional DPs $V_t^j(\mathbf{x})$ and $V_t^{jh}(\mathbf{x})$. While Step 3b removes the redundant inventories, Step 3a computes $V_t(\mathbf{x})$ by recursively using Theorem 5(d), as follows. First note that we can decompose $V_t(\mathbf{x})$ as

$$\begin{aligned}
V_t(\mathbf{x}) &= V_t(\mathbf{x} - \mathbf{e}_{k_1} + \mathbf{e}_{k_2}) + V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1} - 1, 1) \\
&= V_t(\mathbf{x} - 2\mathbf{e}_{k_1} + 2\mathbf{e}_{k_2}) + V_t^{k_1}(x_{k_1} - 1) - V_t^{k_1,k_2}(x_{k_1} - 2, 1) \\
&\quad + V_t^{k_1}(x_{k_1}) - V_t^{k_1,k_2}(x_{k_1} - 1, 1) \\
&= \dots \\
&= V_t(0, \dots, 0, |\mathbf{x}|_{k_2}, x_{k_2+1}, \dots, x_n) + \sum_{x=1}^{|\mathbf{x}|_{k_1}} V_t^{k_1}(x) - \sum_{x=1}^{|\mathbf{x}|_{k_1}-1} V_t^{k_1,k_2}(x, 1),
\end{aligned}$$

Continuing in this manner, we obtain (23) in Step 3a as follows.

$$\begin{aligned}
V_t(\mathbf{x}) &= V_t(0, \dots, 0, |\mathbf{x}|_{k_3}, x_{k_3+1}, \dots, x_n) + \sum_{x=1}^{|\mathbf{x}|_{k_2}} V_t^{k_2}(x) - \sum_{x=1}^{|\mathbf{x}|_{k_2}-1} V_t^{k_2,k_3}(x, 1) \\
&\quad + \sum_{x=1}^{|\mathbf{x}|_{k_1}} V_t^{k_1}(x) - \sum_{x=1}^{|\mathbf{x}|_{k_1}-1} V_t^{k_1,k_2}(x, 1) \\
&= \dots \\
&= \sum_{r=1}^m \sum_{x=1}^{|\mathbf{x}|_{k_r}} V_t^{k_r}(x) - \sum_{j=2}^m \sum_{x=1}^{|\mathbf{x}|_{k_{r-1}}-1} V_t^{k_{r-1},k_r}(x, 1), \quad 1 \leq t \leq T.
\end{aligned}$$

We close this section with a brief discussion on the complexity of our algorithm and the standard backward induction algorithm. Suppose $\kappa = \max_j \{\kappa_j\}$ is the maximum of the individual product inventory levels. Then, solving for the value function $V_t(\mathbf{x})$ for $1 \leq t \leq T$ using backward induction would need an *exponential* number of state evaluations, resulting in a running time of $O(\kappa^n T)$. In contrast, the computational complexity of our algorithm depends on that of the single dimensional DPs V_t^j and V_t^{jh} . Since there are $\frac{n^2+n}{2}$ such DPs, each requiring a running time of $O(n\kappa T)$, the

total running time is $O(n^3\kappa T)$. Therefore, Algorithm 8 is a *polynomial-time* exact algorithm. We can also show that the savings in memory requirements while using this algorithm, over that of the standard backward induction, are similar to the savings in the running time analysis. Clearly, our exact algorithm drastically reduces the computational complexity of the backward induction algorithm and can be implemented for large-sized problems.

6. Pricing of Horizontally Differentiated Products

When customer preferences for products are based on subjective elements such as color, aesthetic appeal, or even schedule convenience (as in the airline example in Section 1), the products in the assortment do not have a universal ordering and are said to be horizontally differentiated. In such an assortment, due to the dispersion of preferences among customers, a product might have positive demand even if its price is substantially higher than other products in the assortment. We denote the pricing problem facing a firm that offers an assortment of horizontally differentiated products as MPDP-H. The multinomial logit (MNL) discrete choice model (see McFadden 1980 and Anderson et al. 1992) to be described next incorporates the choice differences that result from preference dispersion among customers.

6.1. Optimal Prices for MPDP-H

The MNL model adopts a discrete choice approach to modeling consumer behavior. Consider a population of consumers that has a mean sensitivity θ to an attribute, say quality, of the products offered. The scalar parameter μ and the product-specific random variable ε_j with mean 0 and variance 1 capture the uncertainty in the utility that a customer gains when purchasing the product. That is, a consumer who purchases product j , which has a quality index q_j , at price p_{jt} enjoys a utility $u_{jt} = (\theta q_j - p_{jt}) + \mu \varepsilon_j$, where $\mu \varepsilon_j$ reflects the nature of horizontal differentiation between products. As μ increases, the products become more horizontally differentiated, i.e., consumers are influenced more by *taste-related* attributes of the products and less by the price and quality. When ε_j is i.i.d. Gumbel, we obtain the well-known MNL discrete choice model. Using a normalized value of zero for the utility of an outside option, the choice probabilities in the MNL model are:

$$\alpha_j(\mathbf{p}_t) = \frac{e^{(\theta q_j - p_{jt})/\mu}}{1 + \sum_{j=1}^n e^{(\theta q_j - p_{jt})/\mu}}, \quad j = 1, 2, \dots, n. \quad (24)$$

The choice probabilities in (24) allow us to express $G_t(\mathbf{x}, \mathbf{p}_t)$, given in (4), as:

$$G_t(\mathbf{x}, \mathbf{p}_t) = \sum_{j=1}^n \frac{\lambda_t e^{(\theta q_j - p_{jt})/\mu}}{1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt})/\mu}} (p_{jt} - \Delta_{x_j} V_{t-1}(\mathbf{x})), \quad \mathbf{p}_t \in \mathcal{P}_{\mathbf{x}}. \quad (25)$$

While $G_t(\mathbf{x}, \mathbf{p}_t)$ for MNL is not a concave function of \mathbf{p}_t , we can show that it is strictly *quasiconcave* in \mathbf{p}_t using the determinants of its bordered Hessian matrix. Therefore, by Theorem 1, the optimal prices $\mathbf{p}_t(\mathbf{x})$ must satisfy (7). To obtain $\mathbf{h}(\mathbf{p}_t) = -\boldsymbol{\alpha}(\mathbf{p}_t) \left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} \right)^{-1}$, we find the elements of the Jacobian matrix of $\boldsymbol{\alpha}(\mathbf{p}_t)$ are:

$$\frac{\partial \alpha_i(\mathbf{p}_t)}{\partial p_{jt}} = \begin{cases} -\frac{e^{(\theta q_j - p_{jt})/\mu} (1 + \sum_{k \neq j} e^{(\theta q_k - p_{kt})/\mu})}{\mu (1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt})/\mu})^2}, & i = j, \\ \frac{e^{(\theta q_i - p_{it})/\mu} e^{(\theta q_j - p_{jt})/\mu}}{\mu (1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt})/\mu})^2}, & i \neq j. \end{cases}$$

The inverse of this Jacobian matrix is:

$$\left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} \right)^{-1} = -\mu \left(1 + \sum_{k=1}^n e^{\frac{\theta q_k - p_{kt}}{\mu}} \right) \begin{bmatrix} \frac{1 + e^{(\theta q_1 - p_{1t})/\mu}}{e^{(\theta q_1 - p_{1t})/\mu}} & 1 & \dots & 1 \\ 1 & \frac{1 + e^{(\theta q_2 - p_{2t})/\mu}}{e^{(\theta q_2 - p_{2t})/\mu}} & \dots & 1 \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & \dots & \frac{1 + e^{(\theta q_n - p_{nt})/\mu}}{e^{(\theta q_n - p_{nt})/\mu}} \end{bmatrix}$$

Using the Jacobian matrix and its inverse, we derive $h_j(\mathbf{p}_t)$ as:

$$\begin{aligned} h_j(\mathbf{p}_t) &= \mu \left(\frac{e^{(\theta q_1 - p_{1t})/\mu}}{1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt})/\mu}}, \dots, \frac{e^{(\theta q_n - p_{nt})/\mu}}{1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt})/\mu}} \right) \left(1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt})/\mu} \right) \begin{pmatrix} 1 \\ \vdots \\ \frac{1 + e^{(\theta q_j - p_{jt})/\mu}}{e^{(\theta q_j - p_{jt})/\mu}} \\ \vdots \\ 1 \end{pmatrix} \\ &= \mu \left(1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt})/\mu} \right), \quad j = 1, 2, \dots, n, \end{aligned} \quad (26)$$

where $p_{jt} = \infty$ if $x_j = 0$ for some j . In (26), $h_j(\mathbf{p}_t)$ is identical for all j , meaning the marginal values of all products in a given period are the same. This result is consistent with earlier research (Anderson et al. 1992) that reports the optimality of the *uniform pricing* scheme for all horizontally differentiated products in an assortment for a *single period* model. Using (26) and applying Theorem 1, which expresses the optimal price of product j in period t as the sum of $h_j(\mathbf{p}_t)$ (the current marginal value of product j) and $\Delta_{x_j} V_{t-1}(\mathbf{x})$ (the future marginal value of product j), we obtain:

$$p_{jt}(\mathbf{x}) = \mu \left(1 + \sum_{k=1}^n e^{(\theta q_k - p_{kt}(\mathbf{x}))/\mu} \right) + \Delta_{x_j} V_{t-1}(\mathbf{x}). \quad (27)$$

It is not difficult to show that when the firm has surplus inventory of product j ($x_j \geq t$), the future value of the surplus units is zero, that is, $\Delta_{x_j} V_{t-1}(\mathbf{x}) = 0$, if $x_j - t \geq 0$. Consequently, from (27), all products that have surplus inventories should be priced the same (Figure 6 uses a numerical example to demonstrate this property). However, such a uniform pricing scheme is not optimal for products that have inventory shortfalls ($x_j < t$). From (27), we see that any product with an inventory shortfall, regardless of its quality, commands a *higher price* than the uniform price set for the products with inventory surplus. In other words, inventory shortfall of a product translates into a premium charged over the uniform price of products with surplus inventories in the assortment, regardless of their respective qualities. We formally state this result in the following lemma.

Lemma 9 *For a horizontally differentiated product assortment,*

(a) *prices of all products with surplus inventories are the same, i.e., $p_{jt}(\mathbf{x}) = p_{kt}(\mathbf{x})$ if $x_j, x_k \geq t$, and*

(b) *price of a product with inventory shortfall is always higher than that of a product with inventory surplus, i.e., $p_{it}(\mathbf{x}) > p_{jt}(\mathbf{x})$ if $x_i < t$ and $x_j \geq t$.*

The properties stated in Lemma 9 are in sharp contrast with the pricing structure in MPDP-V, where a high quality product is always priced higher than a low quality product, regardless of their respective inventories. Further, Lemma 9 shows that the optimal prices in MPDP-H depend on individual inventory levels of products, instead of the aggregate inventory as in MPDP-V.

Using (3), we can express, $\Delta_t V_t(\mathbf{x})$, the marginal value of time at inventory \mathbf{x} in period t as the weighted average of the time values of different products:

$$\Delta_t V_t(\mathbf{x}) = \sum_{k=1}^n \lambda_t \alpha_k (p_{kt}(\mathbf{x})) (p_{kt}(\mathbf{x}) - \Delta_{x_k} V_{t-1}(\mathbf{x})) = \lambda_t \mu \sum_{k=1}^n e^{(\theta q_k - p_{kt}(\mathbf{x}))/\mu}. \quad (28)$$

The Lambert \mathcal{W} function of x , $\mathcal{W}(x)$, satisfies $\mathcal{W}(x)e^{\mathcal{W}(x)} = x$ and is a non-negative, concave and increasing function for $x \geq 0$. Using the Lambert \mathcal{W} function, we can explicitly express the optimal prices in (27) as:

$$p_{jt}(\mathbf{x}) = \Delta_{x_j} V_{t-1}(\mathbf{x}) + \mu \left\{ 1 + \mathcal{W} \left(\sum_{k=1}^n e^{[\theta q_k - \Delta_{x_k} V_{t-1}(\mathbf{x})]/\mu - 1} \right) \right\}, \quad j = 1, \dots, n. \quad (29)$$

We can also write the value function as follows:

$$V_t(\mathbf{x}) = \lambda_t \mu \mathcal{W} \left(\sum_{k=1}^n e^{[\theta q_k - \Delta_{x_k} V_{t-1}(\mathbf{x})] / \mu - 1} \right) + V_{t-1}(\mathbf{x}). \quad (30)$$

6.2. Interpolation-Based Algorithm for MPDP-H

The structure of MPDP-V allows us to find exact solutions with nominal computational effort through Algorithm 8. However, MPDP-H does not have similar structural properties and hence is computationally intensive. In this section, we present a heuristic method to solve the MPDP-H problem, based on an *approximate dynamic programming* technique (see Powell (2007) for an extensive review of the subject). As mentioned in Section 5.4, the standard backward induction method stores the value function for *all* possible states of a DP in each period. We develop an *interpolation-based* dynamic pricing algorithm to reduce the memory requirements of the standard backward induction procedure. Our method stores the optimal values for only a small subset of the entire state space (which we refer to as *anchor values*) and approximates the values of other states through ad hoc *spline interpolation*. The spline interpolation method uses interpolants that are simple, yet flexible and smooth piecewise polynomial functions (Shikin and Plis 1995). In particular, we use *cubic splines*, which are piecewise cubic polynomials that pass through the stored anchor points, to interpolate the multidimensional value function of MPDP-H. Our algorithm embeds these interpolated values within a standard backward induction method and generates near-optimal solutions for a range of instances. Other researchers (see Johnson et al. (1993) and Trick and Zin (1997)) have used such interpolation-based methods successfully to reduce the dimensionality of large-dimensional dynamic programs.

Consider an MPDP-H problem with starting inventories of $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_n)$ units. We define $\boldsymbol{\Gamma}$ as the state space of the value function $V_t(\mathbf{x})$. In general, in each period t , the i^{th} state variable can assume any integer value in set $\Gamma_i = \{0, 1, \dots, \kappa_i\}$. Let $\hat{\gamma}_i$ be the number of anchor points for the cubic spline interpolation in the i^{th} dimension of the state space and let $\hat{\Gamma}_i \subseteq \Gamma_i$ be the set of these anchor points in this dimension. We denote the space defined by all anchor points as $\hat{\boldsymbol{\Gamma}} = \hat{\Gamma}_1 \times \hat{\Gamma}_2 \times \dots \times \hat{\Gamma}_n \subseteq \boldsymbol{\Gamma}$. Finally, we define $\mathcal{S}(\mathbf{x})$ as the corresponding cubic spline function which passes through the anchor values at $\mathbf{x} \in \hat{\boldsymbol{\Gamma}}$.

In the backward induction step of our algorithm, we compute the approximate value of $V_t(\mathbf{x})$, denoted by $\hat{V}_t(\mathbf{x})$, using the estimates of $\hat{V}_{t-1}(\mathbf{x})$ based on the cubic spline interpolation given by $\mathcal{S}(\mathbf{x})$. The accuracy of $\hat{V}_{t-1}(\mathbf{x})$ clearly depends on the selection of anchor points and associated anchor values. Since the surplus inventory of product i in period $t-1$ does not affect the value function, we need not consider an anchor point that is larger than $\min(x_i, t-1)$ in $\hat{\Gamma}_i$, i.e., $\hat{\Gamma}_i \subseteq \{0, 1, \dots, \min(x_i, t-1)\}$. In our algorithm we include states $0, 1, 2$ and $\min(x_i, t-1)$ in $\hat{\Gamma}_i$ by default as four of the anchor points, and then select another $(\hat{\gamma}_i - 4)$, approximately equidistant, anchor points between 3 and $\min(x_i, t-1) - 1$. After constructing $\hat{\Gamma}$ in this manner, we find the multidimensional cubic spline function passing through $\hat{V}_{t-1}(\mathbf{x})$ for all $x \in \hat{\Gamma}$. Additionally, to compute better estimates for $\hat{V}_t(\mathbf{x})$, we solve MPDP-H optimally for the first two periods using (30), and initialize our algorithm as $\hat{V}_1(\mathbf{x}) = V_1(\mathbf{x})$ and $\hat{V}_2(\mathbf{x}) = V_2(\mathbf{x})$. We describe the interpolation-based MPDP-H algorithm next:

Algorithm 10 (*Interpolation-based MPDP-H Algorithm*)

Step 1 For $t = 1$ and $t = 2$, compute $\hat{V}_t(\mathbf{x}) = V_t(\mathbf{x})$ based on the optimality equation given in (30).

Step 2 For $t = 3$ to T ,

2a Determine the set of anchor points to be used for interpolation:

Initialize $\hat{\Gamma}_i = \{0, 1, 2, \min(\kappa_i, t-1)\}$.

Divide $[3, \min(\kappa_i, t-1) - 1]$ into $\gamma_i - 4$ non-decreasing and approximately equal partitions as follows:

Find $\omega = \frac{\min(\kappa_i, t-1) - 2}{(\gamma_i - 3)}$. For $h = 1$ to $\gamma_i - 4$,

If $2 + h[\omega] \notin \hat{\Gamma}_i$, then let $\hat{\Gamma}_i := \hat{\Gamma}_i \cup \{2 + h[\omega]\}$.

If $2 + h[\omega] \notin \hat{\Gamma}_i$, then let $\hat{\Gamma}_i := \hat{\Gamma}_i \cup \{2 + h[\omega]\}$.

2b Determine the approximate value function $\hat{V}_t(\mathbf{x})$ and approximate prices $\hat{\mathbf{p}}_t(\mathbf{x})$:

Construct the cubic spline function $\mathcal{S}(\mathbf{x})$ passing through $\hat{V}_{t-1}(\mathbf{x})$ where $\mathbf{x} \in \hat{\Gamma}$.

For all $\mathbf{x} \in \Gamma \setminus \hat{\Gamma}$, interpolate $\hat{V}_{t-1}(\mathbf{x}) = \mathcal{S}(\mathbf{x}')$ where $x'_i = \min(x_i, t-1)$.

Compute $\hat{V}_t(\mathbf{x}) = \lambda_t \mu \mathcal{W} \left(\sum_{k=1}^n e^{[\theta q_k - \Delta_{x_k} \hat{V}_{t-1}(\mathbf{x})] / \mu - 1} \right) + \hat{V}_{t-1}(\mathbf{x})$.

Compute $\hat{p}_{jt}(\mathbf{x}) = \Delta_{x_j} \hat{V}_{t-1}(\mathbf{x}) + \mu \left\{ 1 + \mathcal{W} \left(\sum_{k=1}^n e^{[\theta q_k - \Delta_{x_k} \hat{V}_{t-1}(\mathbf{x})] / \mu - 1} \right) \right\}$ for all j .

7. Numerical Results

The numerical tests that we conducted focus on two broad goals. The first goal is to illustrate the behavior of optimal prices for MPDP-V and MPDP-H. The second goal is to explore the effectiveness of the interpolation-based MPDP-H algorithm. In these tests, we assess the effectiveness of the algorithm in two aspects: *proximity to optimal value* and *reduction in memory requirements*.

7.1. Behavior of Optimal Prices of MPDP-V

With the help of a numerical example, we first illustrate the three monotonicity properties of the optimal prices in terms of inventory, time and quality, as discussed in Theorem 6. We then examine the behavior of the price difference $p_{jt}(\mathbf{x}) - p_{j+1,t}(\mathbf{x})$ and demonstrate that it depends only on $|\mathbf{x}|_j$.

We select the parameters in the numerical example as follows. We consider a firm offering customers ($\lambda_t = \lambda = 0.8$) three *vertically differentiated* products, with corresponding quality ratings of $q_1 = 10$, $q_2 = 6$, and $q_3 = 2$. Figure 1 shows the optimal prices of the three products as a function of the inventory level of product 1 in period $t = 40$, with the inventory levels of products 2 and 3 fixed as $x_2 = x_3 = 5$. Note that the prices of all three products are non-increasing in x_1 . Similarly, we can also show that the prices of the three products are non-increasing in x_2 and x_3 (inventory monotonicity). Observe from Figure 1 also that as x_1 increases, the price of each product converges to $\frac{q_j}{2}$, $j = 1, 2, 3$, as one would expect from Theorem 5 (a). In addition, the optimal prices are rank ordered according to their qualities (quality monotonicity). Figure 2 depicts the optimal dynamic prices as a function of the remaining time. Fixing the inventory levels at $x_1 = x_2 = x_3 = 5$, we observe that, as the sales deadline approaches, the firm has to reduce the prices of all products in order to expedite sales (time monotonicity).

Next we use the same example to illustrate Theorem 5, which states that the price difference between any two *adjacent* products j and $j + 1$ depends only on aggregate inventory $|\mathbf{x}|_j$. In our example, it means that the price difference between products 1 and 2, $p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x})$, depends only on x_1 , and the price difference of products 2 and 3, $p_{2t}(\mathbf{x}) - p_{3t}(\mathbf{x})$, depends only on $x_1 + x_2$. Figures 3 and 4 illustrate that for a given value of x_1 , $p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x})$ is constant, and for a given value of $x_1 + x_2$, $p_{2t}(\mathbf{x}) - p_{3t}(\mathbf{x})$ is constant. As one might expect, Figures 3 and 4 also show that the price difference is non-increasing as inventory levels x_1 and x_2 increase.

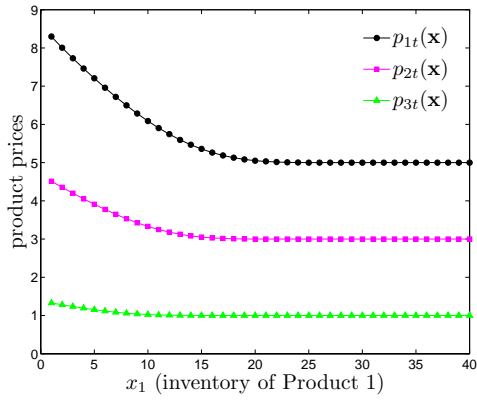


Figure 1 Prices for MPDP-V as a function of x_1 when $x_2 = x_3 = 5$ and $t = 40$

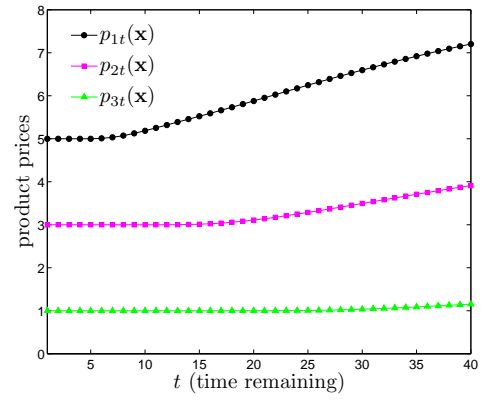


Figure 2 Prices for MPDP-V as a function of t when $x_1 = x_2 = x_3 = 5$

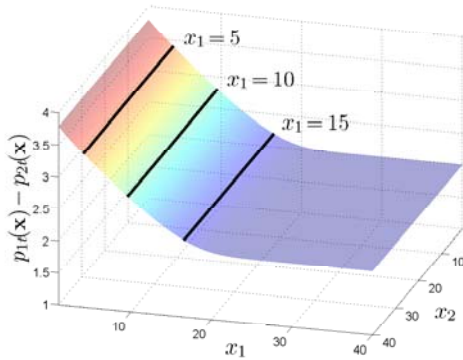


Figure 3 Price difference of products 1 and 2 as a function of x_1 and x_2 when $x_3 = 5$ and $t = 40$

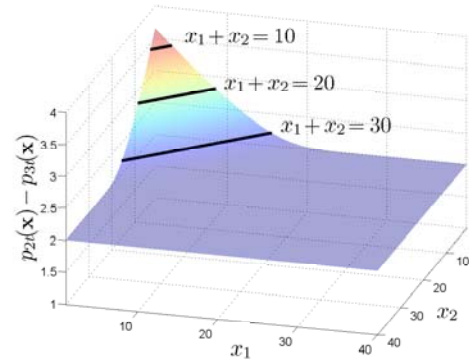


Figure 4 Price difference of products 2 and 3 as a function of x_1 and x_2 when $x_3 = 5$ and $t = 40$

7.2. Behavior of Optimal Prices of MPDP-H

Suppose that the firm in the previous example offers a horizontally differentiated assortment. That is, the three products have features other than quality and, as a result, consumer preferences for these products are dispersed. In addition to retaining quality rating values that we specified earlier, to ensure an appropriate comparison with the MPDP-V example, we let $\theta = 0.5$, corresponding to the mean customer sensitivity in quality in the MPDP-H example. We vary factor μ , which reflects the degree of horizontal differentiation among products, to understand its impact. Figure 5 displays the optimal prices as a function of x_1 in period $t = 40$ at fixed $x_2 = x_3 = 5$, for two different levels of μ . We observe from Figure 5 that while the price of product 1 is non-increasing in its own inventory level, the prices of products 2 and 3 do not demonstrate monotonic behavior as a

function of x_1 , neither when $\mu = 1$ nor when $\mu = 1.5$. In particular, $p_{2t}(\mathbf{x})$ first decreases and then increases in x_1 . Also, the system charges a higher price $p_{3t}(\mathbf{x})$ as product 1 inventory increases. This non-monotonic behavior is in contrast with inventory monotonicity exhibited in MPDP-V and is mainly due to the highly non-linear relationship between joint pricing and consumer choices in this context. Figure 5 also shows that the price is not monotone in terms of quality. For example, for $\mu = 1.5$, price $p_{2t}(\mathbf{x})$ starts to dominate price $p_{1t}(\mathbf{x})$ when x_1 becomes larger. Figure 6 illustrates how the prices of horizontally differentiated products vary over time for fixed inventory levels. Again, the optimal prices do not exhibit the time monotonicity behavior. While the price of product 1 is non-decreasing in t , the prices for both products 2 and 3 first decrease and then increases in t . Furthermore, the optimal prices of all three products, at each setting of μ , converge to the same value as the end of the sales horizon approaches, meaning the *uniform pricing* policy starts to take effect as the time left to sell these products gets shorter. Figures 5 and 6 show sharp contrasts with Figures 1 and 2, in which the pattern of *price differentiation* persists regardless of the inventory levels and time remaining, and the optimal prices always mirror their quality ranking at all times. In both Figures 5 and 6, we observe that the higher value of μ , which signifies a greater degree of horizontal differentiation, allows the firm to charge higher prices. When μ is higher, it results in a larger dispersion in consumer valuations, leading to an increased willingness to pay by consumers.

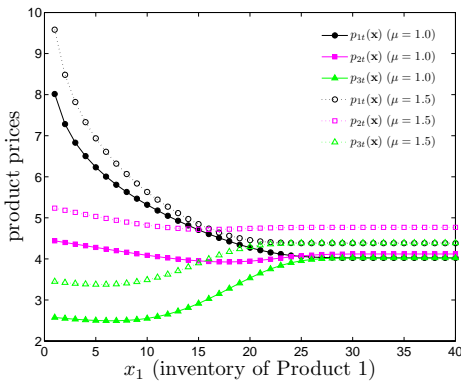


Figure 5 Prices for MPDP-H as a function of x_1 when $x_2 = x_3 = 5$, $t = 40$, $\mu = 1.0$ or 1.5

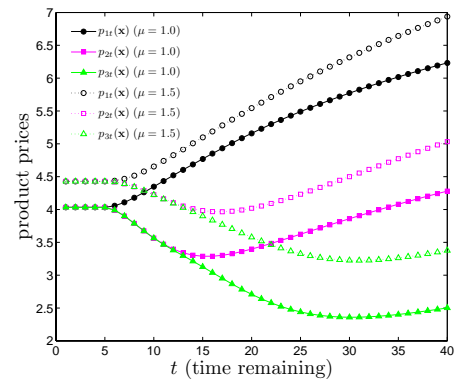


Figure 6 Prices for MPDP-H as a function of t when $x_1 = x_2 = x_3 = 5$, $\mu = 1.0$ or 1.5

Next, we illustrate 9 using another three-product MPDP-H problem with $q_1 = 3$, $q_2 = 2$, $q_3 = 1$, $\mu = 1$, $\theta = 0.5$, and $\lambda_t = 0.8$. Figure 7 depicts the change in product prices over time when $x_1 = 8$, $x_2 = 5$ and $x_3 = 2$. Note that, in region I ($t = 1, 2$), since all three products have surplus inventories, $p_{1t}(\mathbf{x}) = p_{2t}(\mathbf{x}) = p_{3t}(\mathbf{x})$. In region II ($t = 3, 4, 5$), product 1 and 2 have surplus inventories, whereas product 3 has an inventory shortfall. Hence, $p_{1t}(\mathbf{x}) = p_{2t}(\mathbf{x}) <$

$p_{3t}(\mathbf{x})$, although product 3 has the lowest quality. Since, only product 1 has surplus and products 2 and 3 have shortfalls in region III ($t = 6, 7, 8$), we have $p_{1t}(\mathbf{x}) < p_{2t}(\mathbf{x})$ and $p_{1t}(\mathbf{x}) < p_{3t}(\mathbf{x})$. For $t \geq 9$ (region IV), all three products have shortfall inventories, and hence the firm charges different prices.

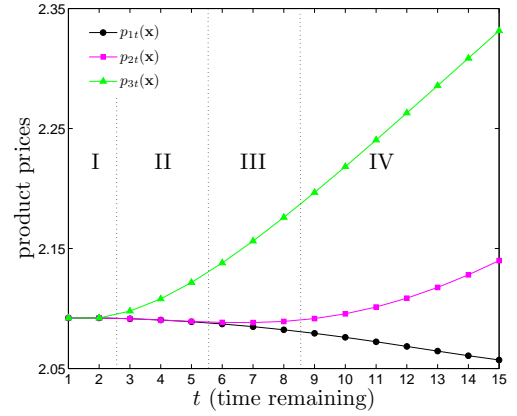


Figure 7 Prices for MPDP-H as a function of t when $x_1 = 8$, $x_2 = 4$, $x_3 = 2$, $\mu = 1$

7.3. Effectiveness of Interpolation-Based MPDP-H Algorithm

To assess the effectiveness of our algorithm, we compare its performance with that of the standard backward induction algorithm over several instances. In each case, we evaluate the algorithm by two performance measures, *approximation error* and *memory requirement reduction*. For a T -period MPDP-H with starting inventory vector $\boldsymbol{\kappa}$, *Approximation Error* measures the absolute value of the gap between the optimal and the algorithm solutions, denoted by $V_T(\boldsymbol{\kappa})$ and $\hat{V}_T(\boldsymbol{\kappa})$, respectively, as a percentage of the optimal solution, i.e., $Approx. \text{ err.} = 100\% \times \frac{|V_T(\boldsymbol{\kappa}) - \hat{V}_T(\boldsymbol{\kappa})|}{V_T(\boldsymbol{\kappa})}$. On the other hand, *memory requirement reduction* expresses the reduction in memory allocation that the interpolation-based algorithm provides, as a percentage of the memory space used by the backward induction method for the optimal solution. Let $Mem(V_T(\boldsymbol{\kappa}))$ and $Mem(\hat{V}_T(\boldsymbol{\kappa}))$ denote physical memories used for finding the optimal and approximate solutions, respectively, to an MPDP-H problem. Then $Memory \text{ req. redn} = 100\% \times \frac{Mem(V_T(\boldsymbol{\kappa})) - Mem(\hat{V}_T(\boldsymbol{\kappa}))}{Mem(V_T(\boldsymbol{\kappa}))}$.

Table 1 records the approximation error and memory requirement reduction for each of the ten

large problem instances that we solved. Each problem instance is characterized by five parameters. To conveniently describe these parameters, we adopt the following notation: $n|(q_1, \dots, q_n)|T|\kappa|\hat{\gamma}$, where n is the number of products, q_j is the quality of product j , T is the number of periods in the sales horizon, κ is the beginning inventory of each product (assuming equal starting inventory for all products), and $\hat{\gamma}$ is the number of anchor points used for each dimension of the state space in the algorithm (assuming equal number of anchor points in each dimension). In all these instances, we set $\lambda_t = \lambda = 0.8$, $\theta = 0.5$, and $\mu = 1$. The memory requirements for the backward induction algorithm to compute the optimal solutions of these problems ranges from 10,201 up to 2,825,761 physical memory locations. The results in Table 1 show that the performance of the interpolation-based algorithm is outstanding, consistently yielding solutions within less than 0.5% of the actual optimal solution, while using less than 1% of the physical memory required by the optimal solution. The excellent solution quality is in spite of a very small number of anchor points, ranging from 5 to 10 points in each dimension.

Problem description					Approx. err.	Memory req. redn
n	(q_1, \dots, q_n)	T	κ	$\hat{\gamma}$	(%)	(%)
2	(10,6)	500	100	10	0.05	99.02
2	(10,6)	500	200	10	0.25	99.75
2	(10,6)	1000	100	10	0.09	99.02
2	(10,6)	1000	200	10	0.03	99.75
3	(10,6,2)	500	50	10	0.09	99.25
3	(10,6,2)	500	100	10	0.06	99.90
3	(10,6,2)	1000	50	10	0.18	99.25
3	(10,6,2)	1000	100	10	0.03	99.90
4	(10,7,4,1)	100	20	5	0.37	99.68
4	(10,7,4,1)	200	40	5	0.40	99.98

Table 1 Effectiveness of the interpolation-based MPDP-H algorithm for large problem instances

8. Conclusions

Advances in information technologies have enabled firms to continuously monitor product availability and customer demand, and adjust prices accordingly to maximize profits. Motivated by applications in industry, in this paper, we study the dynamic pricing problem of a firm that sells

given initial inventories of multiple perishable products over a finite selling season. Since the product demands are linked by consumer choice processes, which are derived in turn from the nature of product differentiation, the firm faces a joint pricing problem. We model this multi-product pricing problem using a stochastic dynamic program, and characterize the structure of optimal prices under a general model of consumer choice. In addition, we present a detailed analysis of the structural properties of the MPDP models for both vertically and horizontally differentiated products that lead to effective solution algorithms.

The results in this paper have the following important managerial implications.

- (1) Vertically and horizontally differentiated products have fundamentally different pricing policy structures. These differences suggest that managers need to select a consumer choice model that is compatible with the specific nature of product differentiation in their applications. The profitability of a firm may be significantly compromised if an inappropriate consumer choice model is used in making pricing decisions.
- (2) The behavior of the optimal pricing policy is driven by consumer preferences for product attributes and availability of product inventories. When products can be (universally) ordered based on their attributes, managers can charge additional premiums for the products ranked higher by consumers. If consumer preferences for products are dispersed, managers can charge additional premiums for the products with scarce inventories.
- (3) The tailored solution algorithms that we develop generate exact or near-optimal solutions with nominal computational effort, allowing managers to readily apply the insights and analysis in this paper to practical-sized problems.

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Proofs of Statements

This appendix contains the proofs of the theorems and lemmas given in the paper.

EC.1. Proof of Theorem 1

To determine the optimal prices, we derive the first order conditions for $G_t(\mathbf{x}, \mathbf{p}_t)$. Since $\Delta_{x_j} V_{t-1}(\mathbf{x})$ is not a function of \mathbf{p}_t , setting the partial derivatives of $G_t(\mathbf{x}, \mathbf{p}_t)$ with respect to p_{jt} to zero gives:

$$\frac{\partial G_t(\mathbf{x}, \mathbf{p}_t)}{\partial p_{jt}} = \sum_{k=1}^n \lambda_t \frac{\partial \alpha_k(\mathbf{p}_t)}{\partial p_{jt}} (p_{kt} - \Delta_{x_k} V_{t-1}(\mathbf{x})) + \lambda_t \alpha_j(\mathbf{p}_t) = 0, \quad j = 1, 2, \dots, n. \quad (\text{EC.1})$$

The conditions in (EC.1) form a system of equations that the optimal price vector \mathbf{p}_t must satisfy.

In matrix form, after eliminating λ_t , this system of equations is:

$$(\mathbf{p}_t - \Delta_{\mathbf{x}} V_{t-1}(\mathbf{x})) \left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} \right) = -\boldsymbol{\alpha}(\mathbf{p}_t). \quad (\text{EC.2})$$

We rewrite equation(EC.2) as:

$$\mathbf{p}_t - \Delta_{\mathbf{x}} V_{t-1}(\mathbf{x}) = -\boldsymbol{\alpha}(\mathbf{p}_t) \left(\frac{\partial \boldsymbol{\alpha}(\mathbf{p}_t)}{\partial \mathbf{p}_t} \right)^{-1} = \mathbf{h}(\mathbf{p}_t). \quad (\text{EC.3})$$

Since $G_t(\mathbf{x}, \mathbf{p}_t)$ is strictly quasi-concave (by assumption), the unique solution, \mathbf{p}_t , of the above expression maximizes function $G_t(\mathbf{x}, \mathbf{p}_t)$. ■

EC.2. Proof of Lemma 3

The first derivative of $G_t(x, p_t)$ with respect to p_t is

$$\begin{aligned} G'_t(x, p_t) &= \lambda_t \alpha'(p_t) (p_t - \Delta_x V_{t-1}(x)) + \lambda \alpha(p_t) \\ &= \lambda_t \alpha'(p_t) \left(p_t + \frac{\alpha(p_t)}{\alpha'(p_t)} - \Delta_x V_{t-1}(x) \right) = \lambda_t \alpha'(p_t) (p_t - h(p_t) - \Delta_x V_{t-1}(x)). \end{aligned} \quad (\text{EC.4})$$

We know that $p_t - h(p_t)$ is a strictly increasing function of p_t . Therefore, there exists a unique value, $p_t(x)$, such that

$$p_t(x) = h(p_t(x)) + \Delta_x V_{t-1}(x). \quad (\text{EC.5})$$

In addition, $\alpha'(p_t) < 0$ since $\alpha(p_t)$ is strictly decreasing in p_t . Therefore, $G'_t(x, p_t) > 0$ for $p_t > p_t(x)$ and $G'_t(x, p_t) < 0$ for $p_t < p_t(x)$. In other words, $G_t(x, p_t)$ strictly increases in p_t for $p_t < p_t(x)$ and strictly decreases in p_t for $p_t > p_t(x)$. Therefore, $G_t(x, p_t)$ is strictly quasi-concave and p_t given in (EC.5) maximizes $G_t(x, p_t)$. ■

EC.3. Proof of Lemma 4

Proof of (a). To show that $\frac{p_{jt}-p_{kt}}{q_j-q_k} \geq \frac{p_{jt}-p_{k+1,t}}{q_j-q_{k+1}}$ for $k = j+1, \dots, n-1$, it is sufficient to show

$(p_{jt}-p_{kt})(q_j-q_{k+1}) - (q_j-q_k)(p_{jt}-p_{k+1,t}) \geq 0$. We have

$$\begin{aligned} & (p_{jt}-p_{kt})(q_j-q_{k+1}) - (q_j-q_k)(p_{jt}-p_{k+1,t}) \\ &= (p_{jt}-p_{kt})(q_j-q_k+q_k-q_{k+1}) - (q_j-q_k)(p_{jt}-p_{kt}+p_{kt}-p_{k+1,t}) \\ &= (p_{jt}-p_{kt})(q_k-q_{k+1}) - (q_j-q_k)(p_{kt}-p_{k+1,t}) \\ &= \sum_{i=j}^{k-1} (p_{it}-p_{i+1,t})(q_k-q_{k+1}) - (q_i-q_{i+1})(p_{kt}-p_{k+1,t}). \end{aligned}$$

Consider the i^{th} term in the above summation. For any $i \leq k-1$,

$$(p_{it}-p_{i+1,t})(q_k-q_{k+1}) - (q_i-q_{i+1})(p_{kt}-p_{k+1,t}) = (q_k-q_{k+1})(q_i-q_{i+1}) \left(\frac{p_{it}-p_{i+1,t}}{q_i-q_{i+1}} - \frac{p_{kt}-p_{k+1,t}}{q_k-q_{k+1}} \right).$$

Clearly, the above expression is nonnegative since $\mathbf{p}_t \in \tilde{\mathcal{P}}_{\mathbf{x}}$.

Proof of (b). We need to show

$$(p_{kt}-p_{jt})(q_{k+1}-q_j) \geq (q_k-q_j)(p_{k+1,t}-p_{jt}) \geq 0.$$

Similarly as in part (a), we have

$$\begin{aligned} & (p_{kt}-p_{jt})(q_{k+1}-q_j) - (q_k-q_j)(p_{k+1,t}-p_{jt}) \\ &= \sum_{i=k+1}^{j-1} ((p_{kt}-p_{k+1,t})(q_i-q_{i+1}) - (q_k-q_{k+1})(p_{it}-p_{i+1,t})) \\ &= \sum_{i=k+1}^{j-1} (q_k-q_{k+1})(q_i-q_{i+1}) \left(\frac{p_{kt}-p_{k+1,t}}{q_k-q_{k+1}} - \frac{p_{it}-p_{i+1,t}}{q_i-q_{i+1}} \right). \end{aligned}$$

Each term of the above summation is nonnegative since $\mathbf{p}_t \in \mathcal{P}$ and $k+1 \leq i \leq j-1$.

Proof of (c). Consider $\alpha_0(\mathbf{p}_t)$ first. A customer will choose an outside option if $\theta q_k - p_{kt} \leq 0$ for all $k = 1, 2, \dots, n$. Since for any $\mathbf{p}_t \in \mathcal{P}$, $\frac{p_{kt}}{q_k}$ is decreasing in k , we have

$$\alpha_0(\mathbf{p}_t) = P \left(\theta \leq \min_{k=1, \dots, n} \left\{ \frac{p_{kt}}{q_k} \right\} \right) = \frac{p_{nt}}{q_n}.$$

Now, consider $\alpha_n(\mathbf{p}_t)$. A customer will buy product n if $\theta q_n - p_{nt} \geq 0$ and $\theta q_n - p_{nt} \geq \theta q_k - p_{kt}$, $k = 1, 2, \dots, n-1$. This translates to

$$\alpha_n(\mathbf{p}_t) = P\left(\frac{p_{nt}}{q_n} \leq \theta \leq \min_{k < n} \left\{ \frac{p_{kt} - p_{nt}}{q_k - q_n} \right\}\right) = \frac{p_{n-1,t} - p_{nt}}{q_{n-1} - q_n} - \frac{p_{nt}}{q_n},$$

where the second equality is due to part (b), which states $\frac{p_{kt} - p_{nt}}{q_k - q_n}$ is decreasing in k .

Next, we derive $\alpha_j(\mathbf{p}_t)$, $1 < j < n$. A customer will buy product j if $\theta q_j - p_{jt} \geq 0$ and $\theta q_j - p_{jt} \geq \theta q_k - p_{kt}$, $j \neq k$. This means

$$\alpha_j(\mathbf{p}_t) = P\left(\max_{k > j} \left\{ \frac{p_{jt}}{q_j}, \frac{p_{jt} - p_{kt}}{q_j - q_k} \right\} \leq \theta \leq \min_{k < j} \left\{ \frac{p_{kt} - p_{jt}}{q_k - q_j} \right\}\right).$$

Since $\mathbf{p}_t \in \mathcal{P}$, we have $\frac{p_{kt}}{p_{jt}} \leq \frac{q_k}{q_j}$ for $k > j$. This gives us

$$\frac{p_{jt} - p_{kt}}{q_j - q_k} = \frac{p_{jt}(1 - \frac{p_{kt}}{p_{jt}})}{q_j(1 - \frac{q_k}{q_j})} \geq \frac{p_{jt}(1 - \frac{q_k}{q_j})}{q_j(1 - \frac{q_k}{q_j})} = \frac{p_{jt}}{q_j}.$$

Furthermore, by part (a), $\frac{p_{jt} - p_{kt}}{q_j - q_k}$ is a decreasing sequence of k for $k > j$, and by part (b), $\frac{p_{kt} - p_{jt}}{q_k - q_j}$ is a decreasing sequence of k for $k < j$, thus

$$\max_{k > j} \left\{ \frac{p_{jt}}{q_j}, \frac{p_{jt} - p_{kt}}{q_j - q_k} \right\} = \frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} \quad \text{and} \quad \min_{k < j} \left\{ \frac{p_{kt} - p_{jt}}{q_k - q_j} \right\} = \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j}$$

Therefore, for $1 < j < n$,

$$\alpha_j(\mathbf{p}_t) = P\left(\frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}} \leq \theta \leq \frac{p_{j-1,t} - p_{jt}}{q_{j-1,t} - q_{jt}}\right) = \frac{p_{j-1,t} - p_{jt}}{q_{j-1} - q_j} - \frac{p_{jt} - p_{j+1,t}}{q_j - q_{j+1}}.$$

Finally, we have

$$\begin{aligned} \alpha_1(\mathbf{p}_t) &= P(\theta q_1 - p_{1t} \geq 0, \theta q_1 - p_{1t} \geq \max_{k \neq 1} \{\theta q_k - p_{kt}\}) \\ &= P\left(\theta \geq \max_{k \neq 1} \left\{ \frac{p_{1t}}{q_1}, \frac{p_{1t} - p_{kt}}{q_1 - q_k} \right\}\right) = 1 - \frac{p_{1t} - p_{2t}}{q_1 - q_2}, \end{aligned}$$

where the last equality follows since $\frac{p_{1t} - p_{kt}}{q_1 - q_k}$ is decreasing in k . ■

EC.4. Proof of Theorem 5

We prove parts (a)–(d) simultaneously, using induction on t . We first establish the base of parts (a)–(d) with $t = 1$. If $|\mathbf{x}|_j \geq t = 1$, we know from (17) that

$$p_{k1}(\mathbf{x}) = \frac{1}{2}(q_k + \Delta_{x_k} V_0(\mathbf{x})), \quad k = 1, 2, \dots, n. \quad (\text{EC.6})$$

Since $\Delta_{x_k} V_0(\mathbf{x}) = 0$, $p_{k1}(\mathbf{x}) = \frac{q_k}{2}$, as long as $x_k > 0$ (otherwise the product is not offered). This proves part (a) with $t = 1$. For $t = 1$, the condition for part (b) is automatically satisfied as long as $|\mathbf{x}|_j \geq 1$, so part (b) holds trivially, in viewing of (EC.6). Finally, parts (c) and (d) hold trivially since, with one period remaining, the optimal price of any product is held at a constant.

We now show, in turn, parts (a)–(d) for period $t + 1$, based on the hypotheses that all four parts are true for t .

Proof of (a). Suppose we have sufficient inventory of the first j types of products to meet the demand during the remaining $t + 1$ periods, i.e., $|\mathbf{x}|_j \geq t + 1$. Then, since $|\mathbf{x}|_j > |\mathbf{x}|_j - 1 \geq t$, our hypothesis for part (b) states that, for any $k \geq j$, $V_t(\mathbf{x} - \mathbf{e}_k) = V_t(\mathbf{x})$. This further implies $\Delta_{x_k} V_t(\mathbf{x}) = 0$, $k \geq j$. Therefore $p_{k,t+1}(\mathbf{x}) = \frac{1}{2}(q_k + \Delta_{x_k} V_t(\mathbf{x})) = \frac{q_k}{2}$, $k \geq j$. The choice probability of product k , $k > j$, is given by

$$\alpha_k(\mathbf{p}_t(\mathbf{x})) = \frac{p_{k-1,t}(\mathbf{x}) - p_{kt}(\mathbf{x})}{q_{k-1} - q_k} - \frac{p_{kt}(\mathbf{x}) - p_{k+1,t}(\mathbf{x})}{q_k - q_{k+1}} = \frac{\frac{q_{k-1}}{2} - \frac{q_k}{2}}{q_{k-1} - q_k} - \frac{\frac{q_k}{2} - \frac{q_{k+1}}{2}}{q_k - q_{k+1}} = 0.$$

This means that as long as we have sufficient inventory from the higher quality products to meet demand, we should set the prices of the lower quality products to the levels so that they will not be chosen by any customer.

Proof of (b). We first show that the pricing decisions with initial states \mathbf{x} and $\mathbf{x} + \mathbf{e}_k$ are the same in period $t + 1$, if $|\mathbf{x}|_j \geq t + 1$. By our hypothesis for part (b),

$$\Delta_{x_i} V_t(\mathbf{x}) = V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_i) = V_t(\mathbf{x} + \mathbf{e}_k) - V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_k) = \Delta_{x_i} V_t(\mathbf{x} + \mathbf{e}_k), \quad i \leq j \leq k.$$

Therefore,

$$p_{i,t+1}(\mathbf{x}) = \frac{1}{2}(q_i + \Delta_{x_i} V_t(\mathbf{x})) = \frac{1}{2}(q_i + \Delta_{x_i} V_t(\mathbf{x} + \mathbf{e}_k)) = p_{i,t+1}(\mathbf{x} + \mathbf{e}_k), \quad i \leq j \leq k.$$

Thus, we set the same prices for the first j products in states \mathbf{x} and $\mathbf{x} + \mathbf{e}_k$ in period $t + 1$. From part (a), we also set the same prices for the next $n - j$ products in states \mathbf{x} and $\mathbf{x} + \mathbf{e}_k$ in period $t + 1$, i.e., $p_{i,t+1}(\mathbf{x}) = p_{i,t+1}(\mathbf{x} + \mathbf{e}_k) = \frac{q_i}{2}$, $j \leq i$. Next, we argue that the pricing policies for the systems with initial states \mathbf{x} and $\mathbf{x} + \mathbf{e}_k$ should be the same from the next period onwards. Note that once the condition $|\mathbf{x}|_j \geq t + 1$ is satisfied, then, since there is at most one request in each period, we always have sufficient inventory from the first j products to meet remaining demand from period t onwards. Therefore, by our hypotheses for part (b), each system will set the same price for each product in each of the remaining $t + 1$ periods. This, of course, implies $V_{t+1}(\mathbf{x}) = V_{t+1}(\mathbf{x} + \mathbf{e}_k)$.

Proof of (c). We need to show, for any $i < i' \leq j$,

$$p_{j,t+1}(\mathbf{x}) = \frac{1}{2}(q_j + \Delta_{x_j} V_t(\mathbf{x})) = \frac{1}{2}(q_j + \Delta_{x_j} V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'})) = p_{j,t+1}(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'}).$$

Without loss of generality, let $|\mathbf{x}|_j < t + 1$, otherwise the result holds trivially, due to part (b). It is sufficient to show, based on our hypothesis for part (c), for $i < i' \leq j$,

$$\begin{aligned} \Delta_{x_j} V_t(\mathbf{x}) &= V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_j) \\ &= V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'}) - V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'} - \mathbf{e}_j) = \Delta_{x_j} V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'}). \end{aligned} \quad (\text{EC.7})$$

The above equation, in essence, states that function $V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'})$ is *independent* of x_j for any $j \geq i' > i$, i.e., it holds as a constant when x_j varies, with other inventory levels fixed. For notation simplicity, denote the four states, \mathbf{x} , $\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'}$, $\mathbf{x} - \mathbf{e}_j$, and $\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'} - \mathbf{e}_j$ by \mathbf{x}^1 , \mathbf{x}^2 , \mathbf{x}^3 and \mathbf{x}^4 , respectively. Note that $\mathbf{x}^1 - \mathbf{e}_j = \mathbf{x}^3$ and $\mathbf{x}^2 - \mathbf{e}_j = \mathbf{x}^4$. Then we can write

$$V_t(\mathbf{x}^1) - V_t(\mathbf{x}^2) = V_t(\mathbf{x}^3) - V_t(\mathbf{x}^4). \quad (\text{EC.8})$$

To prove (EC.8), we use the expression derived in (18). Given \mathbf{x}^i , $i = 1, 2, 3$ and 4 , we have

$$V_t(\mathbf{x}^i) = \lambda_t \left(q_1 - 2p_{1t}(\mathbf{x}^i) + \frac{\sum_{k=1}^{n-1} (p_{kt}(\mathbf{x}^i) - p_{k+1,t}(\mathbf{x}^i))^2}{q_k - q_{k+1}} + \frac{(p_{nt}(\mathbf{x}^i))^2}{q_n} \right) + V_{t-1}(\mathbf{x}_1),$$

Consider the difference of $V_t(\mathbf{x}^1)$ and $V_t(\mathbf{x}^2)$. By our hypothesis for (EC.7), for $k \geq j \geq i' > i$,

$$\Delta_{x_k} V_{t-1}(\mathbf{x}^1) = V_{t-1}(\mathbf{x}) - V_{t-1}(\mathbf{x} - \mathbf{e}_k)$$

$$= V_{t-1}(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'}) - V_{t-1}(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'} - \mathbf{e}_k) = \Delta_{x_k} V_{t-1}(\mathbf{x}^2).$$

This further implies that

$$p_{kt}(\mathbf{x}^1) = p_{kt}(\mathbf{x}) = p_{kt}(\mathbf{x} - \mathbf{e}_i + \mathbf{e}_{i'}) = p_{kt}(\mathbf{x}^2), \quad k \geq j \geq i' > i.$$

Therefore, the difference of $V_t(\mathbf{x}^1)$ and $V_t(\mathbf{x}^2)$ is reduced to

$$\begin{aligned} V_t(\mathbf{x}^1) - V_t(\mathbf{x}^2) &= -2\lambda_t(p_{1t}(\mathbf{x}^1) - p_{1t}(\mathbf{x}^2)) + (V_{t-1}(\mathbf{x}^1) - V_{t-1}(\mathbf{x}^2)) \\ &\quad + \frac{\lambda_t \sum_{k=1}^{j-1} (p_{kt}(\mathbf{x}^1) - p_{k+1,t}(\mathbf{x}^1))^2 - (p_{kt}(\mathbf{x}^2) - p_{k+1,t}(\mathbf{x}^2))^2}{q_k - q_{k+1}}. \end{aligned} \quad (\text{EC.9})$$

We now apply our hypothesis for part (c) to each term in (EC.9). The first term becomes

$$\begin{aligned} p_{1t}(\mathbf{x}^1) - p_{1t}(\mathbf{x}^2) &= \frac{1}{2} (\Delta_{x_1} V_{t-1}(\mathbf{x}^1) - \Delta_{x_1} V_{t-1}(\mathbf{x}^2)) \\ &= \frac{1}{2} (V_{t-1}(\mathbf{x}^1) - V_{t-1}(\mathbf{x}^1 - \mathbf{e}_1) - V_{t-1}(\mathbf{x}^2) + V_{t-1}(\mathbf{x}^2 - \mathbf{e}_1)) \\ &= \frac{1}{2} (V_{t-1}(\mathbf{x}^1 - \mathbf{e}_j) - V_{t-1}(\mathbf{x}^1 - \mathbf{e}_1 - \mathbf{e}_j) - V_{t-1}(\mathbf{x}^2 - \mathbf{e}_j) + V_{t-1}(\mathbf{x}^2 - \mathbf{e}_1 - \mathbf{e}_j)) \\ &= \frac{1}{2} (V_{t-1}(\mathbf{x}^3) - V_{t-1}(\mathbf{x}^3 - \mathbf{e}_1) - V_{t-1}(\mathbf{x}^4) + V_{t-1}(\mathbf{x}^4 - \mathbf{e}_1)) \\ &= \frac{1}{2} (\Delta_{x_1} V_{t-1}(\mathbf{x}^3) - \Delta_{x_1} V_{t-1}(\mathbf{x}^4)) = p_{1t}(\mathbf{x}^3) - p_{1t}(\mathbf{x}^4), \end{aligned}$$

where the third equality uses the hypothesis that functions $V_{t-1}(\mathbf{x}^1) - V_{t-1}(\mathbf{x}^2)$ and $V_{t-1}(\mathbf{x}^1 - \mathbf{e}_1) - V_{t-1}(\mathbf{x}^2 - \mathbf{e}_1)$ are independent of x_j , and the fourth equality uses the fact that $\mathbf{x}^1 - \mathbf{e}_j = \mathbf{x}^3$ and $\mathbf{x}^2 - \mathbf{e}_j = \mathbf{x}^4$. Now, applying the hypothesis for (EC.8) to the second term in (EC.9) yields

$$V_{t-1}(\mathbf{x}^1) - V_{t-1}(\mathbf{x}^2) = V_{t-1}(\mathbf{x}^3) - V_{t-1}(\mathbf{x}^4).$$

Next, consider each expression in the third summation term of (EC.9). For any $k < j$, we have

$$\begin{aligned} p_{kt}(\mathbf{x}^1) - p_{k+1,t}(\mathbf{x}^1) &= \frac{1}{2} (q_k - q_{k+1} + V_{t-1}(\mathbf{x}^1 - \mathbf{e}_{k+1}) - V_{t-1}(\mathbf{x}^1 - \mathbf{e}_k)) \\ &= \frac{1}{2} (q_k - q_{k+1} + V_{t-1}(\mathbf{x}^1 - \mathbf{e}_{k+1} - \mathbf{e}_j) - V_{t-1}(\mathbf{x}^1 - \mathbf{e}_k - \mathbf{e}_j)) \\ &= \frac{1}{2} (q_k - q_{k+1} + V_{t-1}(\mathbf{x}^3 - \mathbf{e}_{k+1}) - V_{t-1}(\mathbf{x}^3 - \mathbf{e}_k)) \\ &= p_{kt}(\mathbf{x}^3) - p_{k+1,t}(\mathbf{x}^3), \end{aligned}$$

where the second equality uses the hypothesis, for $k < k+1 \leq j$, the term $V_{t-1}(\mathbf{x}^1 - \mathbf{e}_{k+1}) - V_{t-1}(\mathbf{x}^1 - \mathbf{e}_k)$ is independent of x_j , and the third equality uses the fact $\mathbf{x}^1 - \mathbf{e}_j = \mathbf{x}^3$. Similarly,

$$p_{kt}(\mathbf{x}^2) - p_{k+1,t}(\mathbf{x}^2) = p_{kt}(\mathbf{x}^4) - p_{k+1,t}(\mathbf{x}^4).$$

Substituting the above identities to the terms in (EC.9), we obtain

$$\begin{aligned} V_t(\mathbf{x}^1) - V_t(\mathbf{x}^2) &= -2\lambda_t(p_{1t}(\mathbf{x}^3) - p_{1t}(\mathbf{x}^4)) + (V_{t-1}(\mathbf{x}^3) - V_{t-1}(\mathbf{x}^4)) \\ &\quad + \frac{\lambda_t \sum_{k=1}^{j-1} (p_{kt}(\mathbf{x}^3) - p_{k+1,t}(\mathbf{x}^3))^2 - (p_{kt}(\mathbf{x}^4) - p_{k+1,t}(\mathbf{x}^4))^2}{q_k - q_{k+1}}. \\ &= V_t(\mathbf{x}^3) - V_t(\mathbf{x}^4). \end{aligned}$$

This establishes (EC.8) and also completes the proof of part (c).

Proof of (d). This part, in fact, is a direct consequence of (EC.7), which states that the difference function $V_{t-1}(\mathbf{x} - \mathbf{e}_{j-1}) - V_{t-1}(\mathbf{x} - \mathbf{e}_j)$ depends only on $(x_1, x_2, \dots, x_{j-1})$. Since

$$p_{j-1,t}(\mathbf{x}) - p_{jt}(\mathbf{x}) = \frac{1}{2}(q_{j-1} - q_j + V_{t-1}(\mathbf{x} - \mathbf{e}_j) - V_{t-1}(\mathbf{x} - \mathbf{e}_{j-1})),$$

the price difference $p_{j-1,t}(\mathbf{x}) - p_{jt}(\mathbf{x})$ also depends only on (x_1, \dots, x_{j-1}) . However, by part (c), price $p_{j-1,t}(\mathbf{x})$ depends on (x_1, \dots, x_{j-1}) only through their sum $|\mathbf{x}|_{j-1}$; similarly, $p_{jt}(\mathbf{x})$ depends on (x_1, \dots, x_j) only through their sum $|\mathbf{x}|_j$. This implies that the price difference of two adjacent products, $j-1$ and j , depends only on $|\mathbf{x}|_{j-1}$. Finally, to see that $p_{j-1,t}(\mathbf{x}) - p_{jt}(\mathbf{x})$ is nonnegative, we note $V_{t-1}(\mathbf{x} - \mathbf{e}_j) \geq V_{t-1}(\mathbf{x} - \mathbf{e}_{j-1})$, since it is always better to have an extra unit of product $j-1$ than to have an extra unit of product j . ■

EC.5. Proof of Theorem 6

Proof of (a)–(c). The proof is by induction on $t + |\mathbf{x}|_n = \ell$. The initial step of induction for $\ell = 1$ is trivially true. Next we prove parts (a)–(c) simultaneously, assuming $t + |\mathbf{x}|_n = \ell + 1$. To facilitate further analysis, we state the following preliminary results first, which follow from the general chain rule of derivatives for the composition of multi-variable functions.

Properties of composition of multivariate functions. Let $g : R^n \rightarrow R$ and $f_j : R^n \rightarrow R$, for $j = 1, 2, \dots, n$. Define $g \circ f(\mathbf{x}) = g(f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x}))$, with $\mathbf{x} = (x_1, x_2, \dots, x_n)$.

(1) If g is non-increasing in each of its arguments and f_j is non-decreasing in each of its arguments, then the composite function $g \circ f$ is non-increasing in each of its arguments.

(2) If g is non-increasing in each of its arguments and f_j is non-increasing in each of its arguments, then the composite function $g \circ f$ is non-decreasing in each of its arguments.

Proof of (a). The statement holds for $t=0$ for any \mathbf{x} , so assume $t > 1$. Let $t+1 + |\mathbf{x}|_n = \ell + 1$.

We now need to show $\Delta_t V_{t+1}(\mathbf{x} + \mathbf{e}_j) \geq \Delta_t V_{t+1}(\mathbf{x})$, for $j = 1, 2, \dots, n$. Let

$$g(\mathbf{p}_t(\mathbf{x})) = \lambda_t \left(q_1 - 2p_{1t}(\mathbf{x}) + \frac{\sum_{k=1}^{n-1} (p_{kt}(\mathbf{x}) - p_{k+1,t}(\mathbf{x}))^2}{q_k - q_{k+1}} + \frac{p_{nt}^2(\mathbf{x})}{q_n} \right). \quad (\text{EC.10})$$

Then $\Delta_t V_{t+1}(\mathbf{x}_j) = V_{t+1}(\mathbf{x}) - V_t(\mathbf{x}) = g(\mathbf{p}_{t+1}(\mathbf{x}))$. It can be seen from the following derivations that $g(\mathbf{p}_t(\mathbf{x}))$ is decreasing in $p_{jt}(\mathbf{x})$:

$$\frac{\partial g(\mathbf{p}_t(\mathbf{x}))}{\partial p_{1t}(\mathbf{x})} = \lambda_t \left(-2 + \frac{2(p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x}))}{q_1 - q_2} \right) = -2\lambda_t \left(1 - \frac{(p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x}))}{q_1 - q_2} \right) \leq 0, \quad (\text{EC.11})$$

$$\frac{\partial g(\mathbf{p}_t(\mathbf{x}))}{\partial p_{kt}(\mathbf{x})} = \lambda_t \left(\frac{-2(p_{k-1,t}(\mathbf{x}) - p_{kt}(\mathbf{x}))}{q_{k-1} - q_k} + \frac{2(p_{kt}(\mathbf{x}) - p_{k+1,t}(\mathbf{x}))}{q_k - q_{k+1}} \right) \leq 0, \quad k = 2, \dots, n, \quad (\text{EC.12})$$

$$\begin{aligned} \frac{\partial g(\mathbf{p}_t(\mathbf{x}))}{\partial p_{nt}(\mathbf{x})} &= \lambda_t \left(-\frac{2(p_{n-1,t}(\mathbf{x}) - p_{nt}(\mathbf{x}))}{q_{n-1} - q_n} - \frac{2p_{nt}(\mathbf{x})}{q_n} \right) \\ &= -2\lambda_t \left(\frac{p_{n-1,t}(\mathbf{x})q_n - p_{nt}(\mathbf{x})q_{n-1}}{q_n(q_{n-1} - q_n)} \right) \leq 0, \end{aligned} \quad (\text{EC.13})$$

where the nonnegativity of each of the above expressions is the result of $\mathbf{p}_t(\mathbf{x}) \in \tilde{\mathcal{P}}_{\mathbf{x}}$, which is defined in (11). Therefore, we need to show that the composite function $g(\mathbf{p}_{t+1}(\mathbf{x}))$ is increasing in x_j , $j = 1, 2, \dots, n$. Based on (EC.11)–(EC.13), we know that $g(\mathbf{p}_t(\mathbf{x}))$ is a decreasing function of each of its arguments. By the properties of the composition of the multivariate functions stated at the beginning of the proof, $g(\mathbf{p}_{t+1}(\mathbf{x}))$ will be increasing in x_j if $p_{k,t+1}(\mathbf{x})$ is decreasing in x_j for all k , i.e., $p_{k,t+1}(\mathbf{x} + \mathbf{e}_j) \leq p_{k,t+1}(\mathbf{x})$ for $k, j = 1, 2, \dots, n$. Also, $p_{k,t+1}(\mathbf{x}) = \frac{1}{2} (q_k + \Delta_{x_k} V_t(\mathbf{x}))$ for $k, j = 1, 2, \dots, n$. Now, by our hypotheses for parts (b) and (c) of the theorem, $\Delta_{x_k} V_t(\mathbf{x})$ is decreasing in x_j . It follows that $p_{k,t}(\mathbf{x})$, a linear transformation of $\Delta_{x_k} V_t(\mathbf{x})$, is decreasing in x_j for all k .

Proof of (b). We need to show, for $t + |\mathbf{x}|_n = l + 1$,

$$\Delta_{x_j} V_t(\mathbf{x}) \geq \Delta_{x_j} V_t(\mathbf{x} + \mathbf{e}_i), \quad i \neq j, \quad i, j = 1, \dots, n. \quad (\text{EC.14})$$

Because a submodular function is symmetric with respect to its two arguments, we assume, without loss of generality, that $i < j$. From Theorem 5 (c), $\Delta_{x_j} V_t(\mathbf{x})$ depends on the values of the first j elements only through their aggregate value. This means

$$\Delta_{x_j} V_t(\mathbf{x}) = \Delta_{x_j} V_t(0, \dots, 0, |\mathbf{x}|_j, x_{j+1}, \dots, x_n), \quad (\text{EC.15})$$

$$\Delta_{x_j} V_t(\mathbf{x} + \mathbf{e}_i) = \Delta_{x_j} V_t(0, \dots, 0, |\mathbf{x}|_j + 1, x_{j+1}, \dots, x_n). \quad (\text{EC.16})$$

Therefore, (EC.14) is equivalent to stating that $V_t(0, \dots, 0, |\mathbf{x}|_j, x_{j+1}, \dots, x_n)$ is a concave function of its j^{th} element. This result will be established in the next part, part (c), of the theorem.

Proof of (c). We prove this result by induction on n , the number of products in the system. If $n = 1$, then by Theorem 2 (b), $V_t(x)$ is a concave function of x in a single-product system. Suppose that in an m -product system, $1 \leq m < n$, the value function $V_t(x_1, \dots, x_m)$ is concave in x_j , $j = 1, 2, \dots, m$. Now consider the value function $V_t(x_1, \dots, x_n)$ in the n -product system. We first show that $V_t(\mathbf{x}) = V_t(x_1, \dots, x_n)$ is a concave function of x_j for $j > 1$, i.e., for $t + |\mathbf{x}|_n = l + 1$,

$$\Delta_{x_j} V_t(\mathbf{x}) \geq \Delta_{x_j} V_t(\mathbf{x} + \mathbf{e}_j), \quad j > 1. \quad (\text{EC.17})$$

Since, by Theorem 5 (c), $\Delta_{x_j} V_t(\mathbf{x})$ depends on the inventories of the first j products only through $|\mathbf{x}|_j$, we obtain

$$\Delta_{x_j} V_t(\mathbf{x}) = \Delta_{x_j} V_t(0, \dots, 0, |\mathbf{x}|_j, x_{j+1}, \dots, x_n), \quad j > 1.$$

Similarly,

$$\Delta_{x_j} V_t(\mathbf{x} + \mathbf{e}_j) = \Delta_{x_j} V_t(0, \dots, 0, |\mathbf{x}|_j + 1, x_{j+1}, \dots, x_n), \quad j > 1.$$

Since $j > 1$, this reduces the problem to the system with $n - j + 1 < n$ products. By our hypothesis, the value function for the system with less than n products is concave in x_j . Therefore, (EC.17) holds with $j > 1$.

Next, we show (EC.17) with $j = 1$. If $x_1 \geq t$, i.e., the inventory of the highest quality product alone is sufficient to meet demand for the remaining periods, then (EC.17) holds trivially, due to part (a) of Theorem 5. On the other hand, if $x_2 = 0$, then the problem reduces to the $n - 1$ product

model, and by our hypothesis, $V_t(x_1, 0, x_2, \dots, x_n)$ is a concave function of x_1 . Therefore, without loss of generality, we let $x_1 < t$ and $x_2 > 0$. We consider the following two cases.

Case 1. In this case, the total inventory of the first two products is sufficient to meet all remaining demand in states $\mathbf{x} - \mathbf{e}_1$, \mathbf{x} and $\mathbf{x} + \mathbf{e}_1$. By part (b) of Theorem 5, the price of product k in all those three states should be set at a constant, $\frac{q_k}{2}$, for $k = 2, \dots, n$. Note that (EC.17) can be rewritten as

$$V_t(\mathbf{x}) - V_t(\mathbf{x} + \mathbf{e}_1) \geq V_t(\mathbf{x} - \mathbf{e}_1) - V_t(\mathbf{x}). \quad (\text{EC.18})$$

Since, by part (a) of the theorem, $V_{t-1}(\mathbf{x} - \mathbf{e}_1) - V_{t-1}(\mathbf{x}) \geq V_t(\mathbf{x} - \mathbf{e}_1) - V_t(\mathbf{x})$, equation (EC.18) will follow if we can show

$$V_t(\mathbf{x}) - V_t(\mathbf{x} + \mathbf{e}_1) \geq V_{t-1}(\mathbf{x} - \mathbf{e}_1) - V_{t-1}(\mathbf{x}),$$

or, equivalently,

$$V_t(\mathbf{x}) - V_{t-1}(\mathbf{x} - \mathbf{e}_1) \geq V_t(\mathbf{x} + \mathbf{e}_1) - V_{t-1}(\mathbf{x}). \quad (\text{EC.19})$$

We can write the LHS of (EC.19) as

$$\begin{aligned} V_t(\mathbf{x}) - V_{t-1}(\mathbf{x} - \mathbf{e}_1) &= \sum_{k=1}^n \lambda \alpha_k(\mathbf{p}_t(\mathbf{x})) h_k(\mathbf{p}_t(\mathbf{x})) + V_{t-1}(\mathbf{x}) - V_{t-1}(\mathbf{x} - \mathbf{e}_1) \\ &= g(\mathbf{p}_t(\mathbf{x})) + \Delta_{x_1} V_{t-1}(\mathbf{x}) \\ &= g(\mathbf{p}_t(\mathbf{x})) + 2p_{1t}(\mathbf{x}) - q_1, \end{aligned} \quad (\text{EC.20})$$

where $g(\mathbf{p}_t(\mathbf{x}))$ is defined in (EC.10). By our hypothesis for part (c),

$$p_{1t}(\mathbf{x}) \geq p_{1t}(\mathbf{x} + \mathbf{e}_1). \quad (\text{EC.21})$$

Also, when $x_1 + x_2 > t$, we know from Theorem 5(b) that the prices of all other products, except product 1, remain constants when the inventory level changes from \mathbf{x} to $\mathbf{x} + \mathbf{e}_1$, i.e., $p_{kt}(\mathbf{x}) = p_{kt}(\mathbf{x} + \mathbf{e}_1) = \frac{q_k}{2}$, $k = 2, \dots, n$. Now, from (EC.11), it is easily seen that

$$\frac{\partial(g(\mathbf{p}_t(\mathbf{x})) + 2p_{1t}(\mathbf{x}) - q_1)}{\partial p_{1t}(\mathbf{x})} = \lambda_t \left(-2 + \frac{2(p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x}))}{q_1 - q_2} \right) + 2 \geq \frac{2\lambda_t(p_{1t}(\mathbf{x}) - p_{2t}(\mathbf{x}))}{q_1 - q_2} \geq 0,$$

that is, $g(\mathbf{p}_t(\mathbf{x})) + 2p_{1t}(\mathbf{x}) - q_1$ is increasing in $p_{1t}(\mathbf{x})$. When x_1 increases, $p_{1t}(\mathbf{x})$ decreases whereas $p_{kt}(\mathbf{x})$ remains a constant for $k = 2, \dots, n$, because of the properties of the composition of the multivariate functions. Therefore, the composite function $g(\mathbf{p}_t(\mathbf{x})) + 2p_{1t}(\mathbf{x}) - q_1$ is decreasing in x_1 .

Case 2. $x_1 + x_2 \leq t$: We can write

$$\begin{aligned} \Delta_{x_1} V_t(\mathbf{x}) &= V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2) + V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2) - V_t(\mathbf{x} - \mathbf{e}_1) \\ &= V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2) + \Delta_{x_2} V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2). \end{aligned}$$

By part (c) of Theorem 5, $\Delta_{x_2} V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2) = \Delta_{x_2} V_t(0, x_1 - 1 + x_2 + 1, x_3, \dots, x_n)$, which is a decreasing function of x_1 , due to our hypothesis that the value function for the $n - 1$ product system is concave with each of its arguments. Therefore, it is sufficient to show $V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2)$ is decreasing in x_1 . But from Theorem 5 (c), $V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2)$ is independent of x_2 . This means that, for $\mathbf{x}' = (x_1, x'_2, x_3, \dots, x_n)$, with $x_1 + x'_2 > t$,

$$\begin{aligned} V_t(\mathbf{x}) - V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2) &= V_t(\mathbf{x}') - V_t(\mathbf{x}' - \mathbf{e}_1 + \mathbf{e}_2) \\ &= V_t(\mathbf{x}') - V_t(\mathbf{x}' - \mathbf{e}_1 + \mathbf{e}_2) + \Delta_{x_2} V_t(\mathbf{x}' - \mathbf{e}_1 + \mathbf{e}_2) \\ &= \Delta_{x_1} V_t(\mathbf{x}'), \end{aligned}$$

where the second equality uses the fact that, when there is sufficient inventory from the first two products to meet all demand in state $\mathbf{x}' - \mathbf{e}_1$, an extra unit of product 2 will not bring any more revenue, that is, $\Delta_{x_2} V_t(\mathbf{x}' - \mathbf{e}_1 + \mathbf{e}_2) = 0$. However, we have shown, in Case 1, that $\Delta_{x_1} V_t(\mathbf{x}')$ is decreasing in x_1 if $x_1 + x'_2 > t$. Therefore, $\Delta_{x_1} V_t(\mathbf{x}) = \Delta_{x_1} V_t(\mathbf{x}') + \Delta_{x_2} V_t(\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2)$ is decreasing in x_1 . This completes the proof of Case 2.

Proof of (d). It can be shown easily that $\Delta_t V_1(\mathbf{x}) \geq \Delta_t V_2(\mathbf{x})$, for any $|\mathbf{x}|_n = 1$. Therefore, we assume part (d) is true when $t + |\mathbf{x}|_n \leq \ell$. Next we need to show, for $t + |\mathbf{x}|_n = \ell + 1$, that $\Delta_t V_t(\mathbf{x}) \geq \Delta_t V_{t+1}(\mathbf{x})$. We have established in (18) that

$$\Delta_t V_t(\mathbf{x}) = \lambda_t \left(q_1 - 2p_{1t}(\mathbf{x}) + \frac{\sum_{k=1}^{n-1} (p_{kt}(\mathbf{x}) - p_{k+1,t}(\mathbf{x}))^2}{q_k - q_{k+1}} + \frac{(p_{nt}(\mathbf{x}))^2}{q_n} \right)$$

Similar to what we did in part (a), let $\Delta_t V_t(\mathbf{x}) = g(\mathbf{p}_t(\mathbf{x}))$. Since we already assume that λ_t is nonincreasing in t , $\mathbf{p}_t(\mathbf{x})$ increasing in t in each of its arguments, i.e., $p_{kt}(\mathbf{x}) \leq p_{k,t+1}(\mathbf{x})$, $k = 1, 2, \dots, n$, would guarantee that $\Delta_t V_t(\mathbf{x}) = g(\mathbf{p}_t(\mathbf{x}))$ is decreasing in t due the properties of the composition of the multivariate functions. We know that $p_{kt}(\mathbf{x}) = \frac{1}{2}(q_k + \Delta_{x_k} V_{t-1}(\mathbf{x}))$ for $k = 1, 2, \dots, n$. Since $t-1 + \sum_{k=1}^n x_k = \ell < \ell+1$, by our hypothesis for part (a), $\Delta_{x_k} V_{t-1}(\mathbf{x})$ is increasing in t for $k = 1, \dots, n$. Therefore, $p_{kt}(\mathbf{x})$, a linear transformation of $\Delta_{x_k} V_{t-1}(\mathbf{x})$, must be increasing in t for $k = 1, 2, \dots, n$. This establishes part (d). ■