Anil K. Chopra: Dynamics of structures

5.7 Analysis of nonlinear response: Newmark´s method (p. 174-179)

5.7 ANALYSIS OF NONLINEAR RESPONSE: NEWMARK'S METHOD

In this section Newmark's method, described in Section 5.4 for linear systems, is extended to nonlinear systems. Although not as simple as the central difference method, it is perhaps the most popular method because of its superior accuracy.

The difference between Eqs. (5.1.3) and (5.1.4) gives an incremental equilibrium equation:

$$m\,\Delta\ddot{u}_i + c\,\Delta\dot{u}_i + (\Delta f_S)_i = \Delta p_i \tag{5.7.1}$$

The incremental resisting force

$$(\Delta f_S)_i = (k_i)_{sec}\,\Delta u_i \tag{5.7.2}$$

where the secant stiffness $(k_i)_{sec}$, shown in Fig. 5.7.1, cannot be determined because u_{i+1} is *not* known. If we make the assumption that over a small time step Δt, the secant

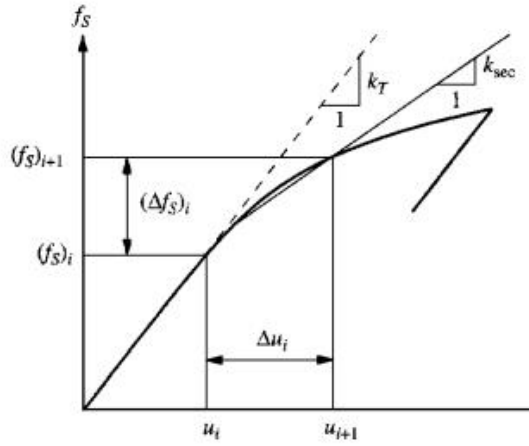

Figure 5.7.1

stiffness $(k_i)_{sec}$ could be replaced by the tangent stiffness $(k_i)_T$ shown in Fig. 5.7.1, then Eq. (5.7.2) could be approximated by

$$(\Delta f_S)_i \simeq (k_i)_T\,\Delta u_i \tag{5.7.3}$$

Dropping the subscript T from $(k_i)_T$ in Eq. (5.7.3) and substituting it in Eq. (5.7.1) gives

$$m\,\Delta\ddot{u}_i + c\,\Delta\dot{u}_i + k_i\,\Delta u_i = \Delta p_i \tag{5.7.4}$$

The similarity between this equation and the corresponding equation for linear systems, Eq. (5.4.12), suggests that the noniterative formulation of Newmark's method presented earlier for linear systems may also be used in the analysis of nonlinear response. All that needs to be done is to replace k in Eq. (5.4.14) by the tangent stiffness k_i to be evaluated at the beginning of each time step. This change implies that step 1.3 of Table 5.4.2 should follow step 2.1. For nonlinear systems step 2.5 and Eq. (5.4.17) would give different values of $\ddot{u}_{i+1}$ and the latter value is preferable because it satisfies equilibrium at time $i + 1$.

This procedure with a constant time step Δt can lead to unacceptably inaccurate results. Significant errors arise for two reasons: (1) the tangent stiffness was used instead of the secant stiffness, and (2) use of a constant time step delays detection of the transitions in the force–deformation relationship.

First, we consider the second source of error, illustrated by the force–deformation relation of Fig. 5.7.2a. Suppose that the displacement at time i, the beginning of a time step, is u_i and the velocity $\dot{u}_i$ is positive (i.e., the displacement is increasing); this is shown by point a. Application of the previously described numerical procedure for the time step results in displacement u_{i+1} and velocity $\dot{u}_{i+1}$ at time $i+1$; this is shown by point b. If $\dot{u}_{i+1}$ is negative, then at some point b' during the time step, the velocity became zero, changed sign, and the displacement started decreasing. In the numerical procedure, if we do not bother to locate b', continue with the computations by starting

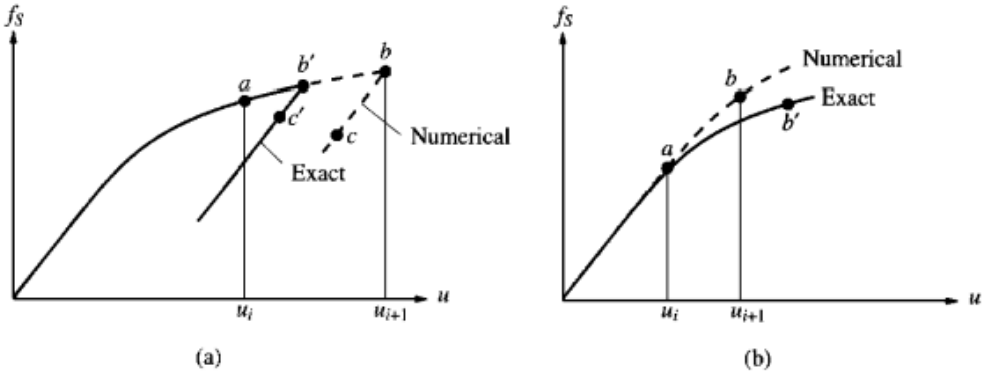

Figure 5.7.2

the next time step at point b, and use the tangent stiffness associated with the unloading branch of the force–deformation diagram, this procedure locates the point c at the end of the next time step with displacement u_{i+2} and negative velocity. On the other hand, if the time instant associated with b'—when the velocity actually became zero—could be determined, computations for the next time step would start with the state of the system at b' and determine the displacement and velocity at the end of the time step, identified as c'. Not locating b' has the effect of overshooting to b and not following the exact path on the force–deformation diagram. These departures from the exact path would occur at each reversal of velocity, leading to errors in the numerical results. A similar problem arises at sharp corners in the force–deformation relationship, as in elastoplastic systems.

These errors could be avoided by locating b' accurately. This could be achieved by retracing the integration over the time interval t_i to t_{i+1} with a smaller time step, say, $\Delta t/4$. Alternatively, an iterative process may be used in which integration is resumed from time i with a step smaller than the full time step, whose size is progressively adjusted so that at the end of such an adjusted time step, the velocity is close to zero.

Now, we return to the first source of error that is associated with the use of tangent stiffness instead of the unknown secant stiffness, and is illustrated by the force–deformation relation of Fig. 5.7.2b. The displacement at time i, the beginning of a time step, is shown as point a. Using the tangent stiffness at a, numerical integration from time i to time $i+1$ leads to the displacement u_{i+1}, identified as point b. If we were able to follow the curve exactly, the result may have been the displacement at b'. This discrepancy accumulating over a series of time steps may introduce significant errors.

These errors can be minimized by using an iterative procedure. The key equation that is solved at each time step in Newmark's method is Eq. (5.4.13), which, modified for nonlinear systems, becomes

$$\hat{k}_i \, \Delta u_i = \Delta \hat{p}_i \tag{5.7.5}$$

where $\Delta \hat{p}_i$ is given by Eq. (5.4.15) and

$$\hat{k}_i = k_i + \frac{\gamma}{\beta \, \Delta t} c + \frac{1}{\beta (\Delta t)^2} m \tag{5.7.6}$$

For convenience in notation we drop the subscript i in k_i and replace it by T to emphasize that this is the tangent stiffness; also, the subscript i is dropped from Δu_i and $\Delta \hat{p}_i$. Equations (5.7.5) and (5.7.6) then become

$$\hat{k}_T \, \Delta u = \Delta \hat{p} \tag{5.7.7}$$

and

$$\hat{k}_T = k_T + \frac{\gamma}{\beta \, \Delta t} c + \frac{1}{\beta (\Delta t)^2} m \tag{5.7.8}$$

Figure 5.7.3a shows a schematic plot of Eq. (5.7.7). The relationship is nonlinear because the tangent stiffness k_T depends on the displacement u and hence the slope $\hat{k}_T$ is not constant. In static analysis of a nonlinear system, $\hat{k}_T = k_T$ and the nonlinearity in $\hat{k}_T$ is the same as in k_T. In dynamic analysis the presence of mass and damping terms in $\hat{k}_T$ decreases the nonlinearity because the constant term $m/\beta(\Delta t)^2$ for typical values of Δt is usually much larger than k_T.

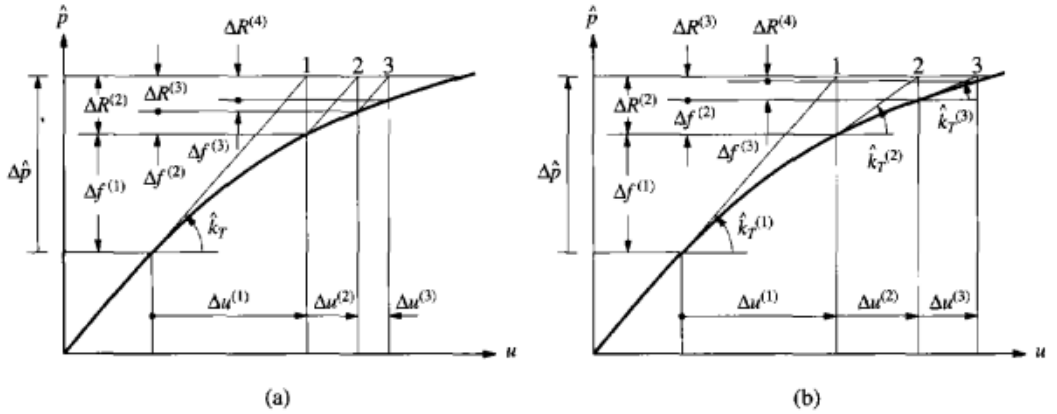

Figure 5.7.3 Iteration within a time step for nonlinear systems: (a) modified Newton–Raphson iteration; (b) Newton–Raphson iteration.

The iterative procedure is described next with reference to Fig. 5.7.3a. The first iterative step is the application of Eq. (5.7.7) in the procedure described previously:

$$\hat{k}_T \, \Delta u^{(1)} = \Delta \hat{p} \tag{5.7.9}$$

to determine $\Delta u^{(1)}$ (corresponding to point b in Fig. 5.7.2b), the first approximation to the final Δu (corresponding to point b' in Fig. 5.7.2b). Associated with $\Delta u^{(1)}$ is the true force $\Delta f^{(1)}$, which is less than $\Delta \hat{p}$, and a residual force is defined: $\Delta R^{(2)} = \Delta \hat{p} - \Delta f^{(1)}$.

The additional displacement $\Delta u^{(2)}$ due to this residual force is determined from

$$\hat{k}_T \, \Delta u^{(2)} = \Delta R^{(2)} = \Delta \hat{p} - \Delta f^{(1)} \tag{5.7.10}$$

This additional displacement is used to find a new value of the residual force, and the process is continued until convergence is achieved. This iterative process for the time step i to $i+1$, summarized in Table 5.7.1, is known as the modified Newton–Raphson method.

TABLE 5.7.1 MODIFIED NEWTON–RAPHSON ITERATION

1.0 *Initialize data.*

$$u_{i+1}^{(0)} = u_i \qquad f_S^{(0)} = (f_S)_i \qquad \Delta R^{(1)} = \Delta \hat{p}_i \qquad \hat{k}_T = \hat{k}_i$$

2.0 *Calculations for each iteration,* $j = 1, 2, 3, \ldots$

2.1 Solve: $\hat{k}_T \, \Delta u^{(j)} = \Delta R^{(j)} \Rightarrow \Delta u^{(j)}$.

2.2 $u_{i+1}^{(j)} = u_{i+1}^{(j-1)} + \Delta u^{(j)}$.

2.3 $\Delta f^{(j)} = f_S^{(j)} - f_S^{(j-1)} + (\hat{k}_T - k_T) \, \Delta u^{(j)}$.

2.4 $\Delta R^{(j+1)} = \Delta R^{(j)} - \Delta f^{(j)}$.

3.0 *Repetition for next iteration.* Replace j by $j+1$ and repeat calculation steps 2.1 to 2.4.

The iterative process is terminated after ℓ iterations when the incremental displacement $\Delta u^{(\ell)}$ becomes small enough compared to the current estimate of $\Delta u = \sum_{j=1}^{\ell} \Delta u^{(j)}$; that is,

$$\frac{\Delta u^{(\ell)}}{\Delta u} < \epsilon$$

Then the displacement increment over the time step i to $i+1$ is given by

$$\Delta u_i = \sum_{j=1}^{\ell} \Delta u^{(j)} \tag{5.7.11}$$

This is an accurate value of Δu_i that replaces the one obtained without iteration from Eq. (5.7.5); the latter is the same as $\Delta u^{(1)}$ obtained after one iteration.

With Δu_i known, the rest of the computation proceeds as before; in particular, $\Delta \ddot{u}_i$ and $\Delta \dot{u}_i$ are determined from Eqs. (5.4.10) and (5.4.11), respectively. Table 5.7.2 summarizes the time-stepping solution as it might be implemented on the computer.

The original Newton–Raphson method converges more rapidly, as illustrated in Fig. 5.7.3b, than the above-described iterative process at the expense of additional computation. This improved convergence is achieved by using in each iteration the current tangent stiffness $k_T^{(j)}$ and the associated value of $\hat{k}_T^{(j)}$ from Eq. (5.7.8) instead of k_T and $\hat{k}_T$. At each iteration the residual force vector $\Delta R^{(j)}$ is now smaller, as seen by comparing parts (a) and (b) of the figure, and the process will converge in fewer iterations.

TABLE 5.7.2 NEWMARK'S METHOD: NONLINEAR SYSTEMS

Special cases

(1) Average acceleration method ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$)

(2) Linear acceleration method ($\gamma = \frac{1}{2}$, $\beta = \frac{1}{6}$)

1.0 *Initial calculations*

1.1 $\ddot{u}_0 = \dfrac{p_0 - c\dot{u}_0 - (f_S)_0}{m}$.

1.2 Select Δt.

1.3 $a = \dfrac{1}{\beta \Delta t} m + \dfrac{\gamma}{\beta} c$; and $b = \dfrac{1}{2\beta} m + \Delta t \left(\dfrac{\gamma}{2\beta} - 1 \right) c$.

2.0 *Calculations for each time step, i*

2.1 $\Delta \hat{p}_i = \Delta p_i + a\dot{u}_i + b\ddot{u}_i$.

2.2 Determine the tangent stiffness k_i.

2.3 $\hat{k}_i = k_i + \dfrac{\gamma}{\beta \Delta t} c + \dfrac{1}{\beta (\Delta t)^2} m$.

2.4 Solve for Δu_i from $\hat{k}_i$ and $\Delta \hat{p}_i$ using the iterative procedure of Table 5.7.1.

2.5 $\Delta \dot{u}_i = \dfrac{\gamma}{\beta \Delta t} \Delta u_i - \dfrac{\gamma}{\beta} \dot{u}_i + \Delta t \left(1 - \dfrac{\gamma}{2\beta} \right) \ddot{u}_i$.

2.6 $\Delta \ddot{u}_i = \dfrac{1}{\beta (\Delta t)^2} \Delta u_i - \dfrac{1}{\beta \Delta t} \dot{u}_i - \dfrac{1}{2\beta} \ddot{u}_i$.

2.7 $u_{i+1} = u_i + \Delta u_i$, $\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i$, $\ddot{u}_{i+1} = \ddot{u}_i + \Delta \ddot{u}_i$.

3.0 *Repetition for the next time step.* Replace i by $i + 1$ and implement steps 2.1 to 2.7 for the next time step.