An Euler beam element (Figure 2) is defined as a beam with two nodes, a uniform geometry (area A and momentum of inertia I) and constant material properties (density and E-modulus). One element has 3 degrees of freedom for each end point (axial and perpendicular translation as well as rotation), but the axial translation will be ignored, since only bending is considered. Hence, 4 dofs remain for one element.

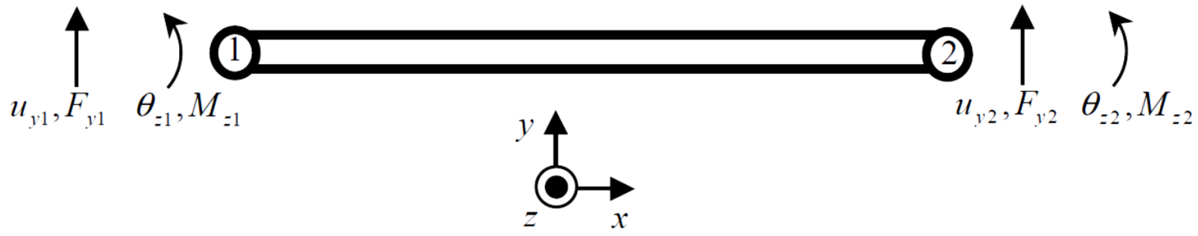

Figure 1: Euler beam element

The stiffness matrix for each element of length l is

$$f^e = K^e u^e; \quad f^e = \begin{bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \end{bmatrix}; \quad u^e = \begin{bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{bmatrix}; \quad K^e = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ & 4l^2 & -6l & 2l^2 \\ & & 12 & -6l \\ sym & & & 4l^2 \end{bmatrix}$$

And the corresponding mass matrix is

$$f^e = M^e \ddot{u}^e; \quad f^e = \begin{bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \end{bmatrix}; \quad \ddot{u}^e = \begin{bmatrix} \ddot{u}_{y1} \\ \ddot{\theta}_{z1} \\ \ddot{u}_{y2} \\ \ddot{\theta}_{z2} \end{bmatrix}; \quad M^e = \frac{\rho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ & 4l^2 & 13l & -3l^2 \\ & & 156 & -22l \\ sym & & & 4l^2 \end{bmatrix}$$

Figure 2: Beam element sections fort he given rotor

The general equation of is $M\ddot{\underline{q}} + K\underline{q} = \underline{0}$ with $q_{i=1..9} = [u_i, \theta_i]^T$. Using the Euler beam elements, M and K are obtained by superposition of the corresponding element matrices M^e and K^e. Additionally, the bearing stiffness has to be added to the vertical direction at the first and the last node. The eigenfrequencies and mode shapes can then be calculated from the eigenvalues of the dynamic equation, thus

$$\left(\lambda^2 M + K\right)\underline{q} = \underline{0}$$

For numerical reasons, the second order differential equation of motion has been reduced to a set of first order differential equations by using the state space formulation.

$$\underline{C}\,\underline{\dot{y}} + \underline{D}\,\underline{y} = 0 \qquad \underline{C} = \begin{bmatrix} \underline{0} & \underline{M} \\ \underline{M} & \underline{0} \end{bmatrix} \qquad \underline{D} = \begin{bmatrix} \underline{K} & \underline{0} \\ \underline{0} & -\underline{M} \end{bmatrix} \qquad \underline{y} = \left[\underline{q}, \underline{\dot{q}}\right]^T$$

This results in 2 dofs * 11 nodes * 2 = 44 eigenvalues and eigenvectors. The first 22 eigenvalues and eigenvectors are the solutions to the equation of motion, where the last 22 eigenvectors are a linear combination of the eigenvalue and the first 22 eigenvectors. The angular eigenfrequencies can be calculated from the complex eigenvalues, by taking the absolute value, so the eigenfrequencies are

$$\underline{f} = \frac{|\underline{\lambda}|}{2\pi}$$

All eigenvectors are normalized to amplitude 1, so that the mode shapes can be visualized. However, the shapes have not been fitted to a smooth curve.

Note that this approach only gives a rough estimation of the eigenfrequencies. But increasing the number of elements in this method does not yield more accurate results as this reduces the length to diameter ratio. The lower this ratio, the less accurate is this approach.