Paper from 13th IFAC World Congress, San Francisco, CA, 1996

Swinging Up a Pendulum by Energy Control*

K. J. Åström† and K. Furuta‡

†Department of Automatic Control
Lund Institute of Technology, Box 118, Lund, Sweden
Phone +46-46-2228781, Fax +46-46-138118, email: kja@Control.LTH.SE
Corresponding Author

‡Department of Control and Systems Engineering
Tokyo Institute of Technology, Tokyo, Japan

Abstract

Properties of simple strategies for swinging up an inverted pendulum are discussed. It is shown that the behavior critically depends on the ratio of the maximum acceleration of the pivot to the acceleration of gravity. A comparison of energy based strategies with minimum time strategy gives interesting insights into the robustness of minimum time solutions.

Keywords:

Inverted pendulum, swing-up, energy control, minimum time control.

1. Introduction

Inverted pendulums have been classic tools in the control laboratories since the 1950s. They were originally used to illustrate ideas in linear control such as stabilization of unstable systems, see e.g. Schaefer and Cannon (1967), Mori *et al.* (1976), Maletinsky *et al.* (1981), and Meier *et al.* (1990). Because of their nonlinear nature pendulums have maintained their usefulness and they are now used to illustrate many of the ideas emerging in the field of nonlinear control. Typical examples are feedback stabilization, variable structure control (Yamakita and Furuta (1992)), passivity based control (Fradkov *et al.* (1995)), backstepping and forwarding (Krstić *et al.* (1994)), nonlinear observers (Eker and Åström (1996)), friction compensation (Abelson (1996)), and nonlinear model reduction. Pendulums have also been used to illustrate task oriented control such as swinging up and catching the pendulum, see Furuta and Yamakita (1991),

* This project was supported by the Swedish Research Council for Engineering Science under contract 95-759 and Nippon Steel Corporation Japan.

Furuta *et al.* (1992), Wiklund *et al.* (1993), Yamakita *et al.* (1993), Yamakita *et al.* (1994), Spong (1995), Spong and Praly (1995), Chung and Hauser (1995), Yamakita *et al.* (1995), Wei *et al.* (1995), Bortoff (1996), Lin *et al.* (1996), Fradkov and Pogromsky (1996), Fradkov *et al.* (1997) Lozano and Fantoni (1998). Pendulums are also excellently suited to illustrate hybrid systems, (Guckenheimer (1995)), Åström (1998) and control of chaotic systems, (Shinbrot *et al.* (1992)).

In this paper we will investigate some properties of the simple strategies for swinging up the pendulum based on energy control. The position and the velocity of the pivot are not considered in the paper. The main results is that the global behavior of the swing up is completely characterized by the ratio n of the maximum acceleration of the pivot and the acceleration of gravity. For example it is shown that one swing is sufficient if n is larger than $4/3$. The analysis also gives insight into the robustness of minimum time swing up in terms of energy overshoot.

The ideas of energy control can be generalized in many different ways. Spong (1995) and Chung and Hauser (1995) have shown that it can be used to also control the position of the pivot. An application to multiple pendulums is sketched in the end of the paper. The ideas have been applied to many different laboratory experiments, see e.g. Iwashiro *et al.* (1996), Eker and Åström (1996) and Åström *et al.* (1995).

2. Preliminaries

Consider a single pendulum. Let its mass be m and let the moment of inertia with respect to the pivot point be J. Furthermore let l be the distance from the pivot to the center of mass. The angle between the vertical and the pendulum is θ, where θ is positive in the clock-wise direction. The acceleration of gravity is g and the acceleration of the pivot is u. The acceleration u is positive if it is in the direction of the positive x-axis. The equation of motion for the pendulum is

$$J\ddot{\theta} - mgl\sin\theta + mul\cos\theta = 0. \tag{1}$$

The system has two state variables, the angle θ and the rate of change of the angle $\dot{\theta}$. It is natural to let the state space be a cylinder. In this state space the system has two equilibria corresponding to $\theta = 0$, $\dot{\theta} = 0$, and $\theta = \pi$, $\dot{\theta} = 0$. If the state space is considered as R^2 there are infinitely many equilibria. There are many deeper differences between the choice of states.

The model given by Equation (1) is based on several assumptions: friction has been neglected and it has been assumed that the pendulum is a rigid body. It has also been assumed that there is no limitation on the velocity of the pivot. The energy of the uncontrolled pendulum ($u = 0$) is

$$E = \frac{1}{2}J\dot{\theta}^2 + mgl(\cos\theta - 1). \tag{2}$$

It is defined to be zero when the pendulum is in the upright position. The model given by Equation (1) has four parameters: the moment of inertia J, the mass m, the length l, and the acceleration of gravity g. Introduce the maximum acceleration of the pivot

$$u_{max} = \max|u| = ng. \tag{3}$$

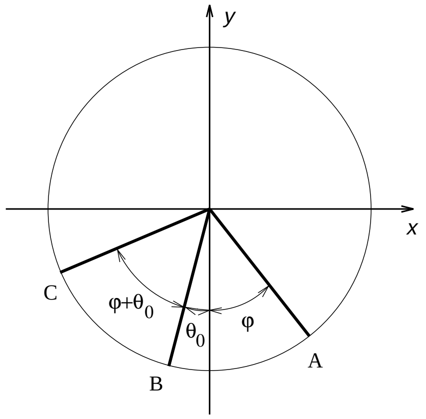

Figure 1 Geometric illustration of a simple swing-up strategy. The origin of the coordinate system is called O.

Introduce the normalized variables $\omega_0 = \sqrt{mgl/J}t$, $\tau = \sqrt{mgl/J}t = \omega_0 t$ and $v = u/g$. The equation of motion (1) then becomes

$$\frac{d^2\theta}{d\tau^2} - \sin\theta + v\cos\theta = 0,$$

where $|v| \le n$. The normalized total energy of the uncontrolled system ($v = 0$) is

$$E_n = \frac{E}{mgl} = \frac{1}{2}\Big(\frac{d\theta}{d\tau}\Big)^2 + \cos\theta - 1. \tag{4}$$

The system is thus characterized by two parameters only, the natural frequency for small oscillations $\omega_0 = \sqrt{mgl/J}$ and the normalized maximum acceleration of the pendulum $n = u_{max}/g$. The model given by Equation (1) is locally controllable when $\theta \ne \pi/2$, i.e. for all states except when the pendulum is horizontal.

A Simple Swing-up Strategy

Before going into technicalities we will discuss a simple strategy for swinging up the pendulum. Consider the situation shown in Figure 1 where the pendulum starts with zero velocity at the point A. Let the pivot accelerate with maximum acceleration ng to the right. The gravity filed seen by an observer fixed to the pivot has the direction OB where $\theta = \arctan n$, and the magnitude $g\sqrt{1+n^2}$. The pendulum then swings symmetrically around OB. The velocity is zero when it reaches the point C where the angle is $\varphi + 2\theta_0$. The pendulum thus increases its swing angle by $2\theta_0$ for each reversal of the velocity. The simple strategy we have described can be considered as a simple way of pumping energy into the pendulum. In the next sections we will elaborate on this simple idea.

3. Energy Control

Many tasks can be accomplished by controlling the energy of the pendulum instead of controlling its position and velocity directly, see Wiklund *et al.* (1993). For example one way to swing the pendulum to the upright position is to give it an energy that corresponds to the upright position. This corresponds to the trajectory

$$E = \frac{1}{2}J(\dot{\theta})^2 + mgl(\cos\theta - 1) = 0,$$

which passes through the unstable equilibrium at the upright position. A different strategy is used to catch the pendulum as it approaches the equilibrium. Such a strategy can also catch the pendulum even if there is an error in the energy control so that the constant energy strategy does not pass through the desired equilibrium, see Åström (1999).

The energy E of the uncontrolled pendulum is given by Equation (2). To perform energy control it is necessary to understand how the energy is influenced by the acceleration of the pivot. Computing the derivative of E with respect to time we find

$$\frac{dE}{dt} = J\dot{\theta}\ddot{\theta} - mgl\dot{\theta}\sin\theta = -mul\dot{\theta}\cos\theta, \tag{5}$$

where Equation (1) has been used to obtain the last equality. Equation (5) implies that it is easy to control the energy. The system is simply an integrator with varying gain. Controllability is lost when the coefficient of u in the right hand side of (5) vanishes. This occurs for $\dot{\theta} = 0$ or $\theta = \pm\pi/2$, i.e., when the pendulum is horizontal or when it reverses its velocity. Control action is most effective when the angle θ is 0 or π and the velocity is large. To increase energy the acceleration of the pivot u should be positive when the quantity $\dot{\theta}\cos\theta$ is negative. A control strategy is easily obtained by the Lyapunov method. With the Lyapunov function $V = (E - E_0)^2/2$, and the control law

$$u = k(E - E_0)\dot{\theta}\cos\theta, \tag{6}$$

we find that

$$\frac{dV}{dt} = -mlk\big((E - E_0)\dot{\theta}\cos\theta\big)^2.$$

The Lyapunov function decreases as long as $\dot{\theta} \neq 0$ and $\cos\theta \neq 0$. Since the pendulum can not maintain a stationary position with $\theta = \pm\pi/2$ the strategy (6) drives the energy towards its desired value E_0. There are many other control laws that accomplishes this. To change the energy as fast as possible the magnitude of the control signal should be as large as possible. This is achieved with the control law

$$u = ng\,\mathrm{sign}\Big((E - E_0)\dot{\theta}\cos\theta\Big), \tag{7}$$

which drives the function $V = |E - E_0|$ to zero and E towards E_0. The control law (7) may result in chattering. This is avoided with the control law

$$u = \mathrm{sat}_{ng}\Big(k(E - E_0)\mathrm{sign}(\dot{\theta}\cos\theta)\Big), \tag{8}$$

where sat_{ng} denotes a linear function which saturates at ng. The strategy (8) behaves like the linear controller (6) for small errors and like the strategy (7) for large errors. Notice that the function sign is not defined when its argument is zero. If the value is defined as zero the control signal will be zero when the pendulum is at rest or when it is horizontal. If the pendulum starts at rest in the downward position the strategies (7), (6) and (8) all give $u = 0$ and the pendulum will remain in the downward position.

The parameter n is crucial because it gives the maximum control signal and thus the maximum rate of energy change, compare with Equation (5). Parameter n drastically influences the behavior of the swing up as will be shown later. Parameter k determines the region where the strategy behaves linearly. For large values of k the strategy (8) is arbitrarily close to the strategy that gives the maximum increase or decrease of the energy. In practical experiments the parameter is determined by the noise levels on the measured signals.

4. Swing Up Behaviors

We will now discuss strategies for bringing the pendulum to rest in the upright position. The analysis will be carried out for the strategy given by Equation (7). The sign function in Equation (7) is defined to be $+1$ when the argument is zero. The energy of the pendulum given by Equation (2) is defined so that it is zero in the stable upright position and $-2mgl$ in the downward position. With these conventions the acceleration is always positive when the pendulum starts at rest in the downward position.

Energy control with $E_0 = 0$ gives the pendulum the desired energy. The motion approaches the manifold where the energy is zero. This manifold contains the desired equilibrium. With energy control the equilibrium is an unstable saddle. It is necessary to use another strategy to catch and stabilize the pendulum in the upright position. In Malmborg *et al.* (1996) it is shown how to design suitable hybrid strategies. Before considering the details we will make a taxonomy of the different strategies. We will do this by characterizing the gross behavior of the pendulum and the control signal during swing-up. The number of swings the pendulum makes before reaching the upright position is used as the primary classifier and the number of switches of the control signal as a secondary classifier. It turns out that the gross behavior is entirely determined by the maximum acceleration of the pivot ng. The behavior during swing up is simple for large values of ng and becomes more complicated with decreasing values of ng.

Single-Swing Double-Switch Behavior

There are situations where the pendulum swings in such a way that the angle increases or decreases monotonically. This is called the single-swing behavior. If the available acceleration is sufficiently large, the pendulum can be swung up simply by using the maximum acceleration until the desired energy is obtained and then setting acceleration to zero. With this strategy the control signal switches from zero to its largest value and then back to zero again. This motivates the name of the strategy.

To find the strategy we will consider a coordinate system fixed to the pivot of the pendulum and regard the force due to the acceleration of the pivot as an external force. In this coordinate system the center of mass of the pendulum moves along a circular path with radius l. It follows from Equation (8) that the desired energy must be reached before the pendulum is horizontal.

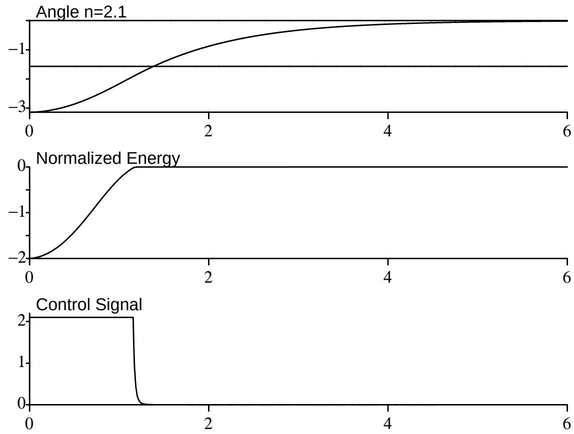

Figure 2 Simulation of a single-swing double-switch strategy. The parameters are $n = 2.1$, $\omega_0 = 1$ and $k = 100$.

The energy supplied to a mass when it is moved from a to b by a force F is

$$W_{ab} = \int_a^b F dx. \tag{9}$$

To swing up the pendulum with only two switches of the control signal the pendulum must have obtained the required energy before the pendulum is horizontal. In a coordinate system fixed to the pivot the center of mass of the pendulum has moved the distance l when it becomes horizontal. The horizontal force is mng and its energy has thus been increased by $mngl$. The energy required to swing up the pendulum is $2mgl$ and we thus find that the maximum acceleration must be at least $2g$ for single-swing double-switch behavior. If the acceleration is larger than $2g$ the acceleration will be switched off when the pendulum angle has changed by θ^*. The center of the mass has moved the distance $l \sin\theta^*$ and the energy supplied to the pendulum is $nmgl \sin\theta^*$. Equating this with $2mgl$ gives $\sin\theta^* = 2/n$.

EXAMPLE 1—SIMULATION OF SSDS BEHAVIOR
The single-swing double-switch strategy is illustrated in Figure 2 which shows the angle, the normalized energy, and the control signal. The simulation is made using the normalized model with $\omega_0 = 1$ and the control law given by Equation (8) with $n = 2.1$ and $k = 100$. With this value of k the behavior is very close to a pure switching strategy. Notice that it is required to have $n \geq 2$ to have the single-swing double switch behavior for pure switching. Slightly larger values of n are required with the control law (8). For the simulations in Figure 2 we used $n = 2.1$. For a pure switching strategy (7) the control signal is switched to zero when the pendulum is 17.8° below the horizontal line. □

Single-Swing Triple-Switch Behavior

To obtain the single-swing double-switch behavior the pendulum must be given sufficient energy before it reaches the horizontal position. In the previous section we found that the condition is $n > 2$. It is possible to have single-swing behavior for smaller values of n but the control signal must then switch three times because the acceleration must be reversed when the pendulum is horizontal. Since the pendulum must reach the horizontal in one swing we must still require

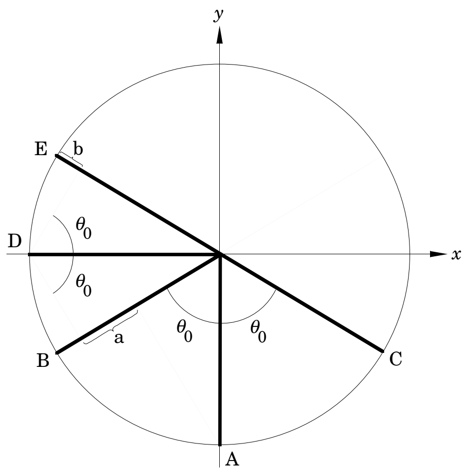

Figure 3 Diagram used to explain the single-swing, triple-switch behavior. The origin of the coordinate system is called O.

that $n > 1$. To find out how much larger it has to be we will consider the situation illustrated in Figure 3. The pendulum starts at rest at position A. The pivot is then accelerated ng in the direction of the positive x-axis. An observer fixed to the pivot sees a gravitational field in the direction OB with the strength $w = g\sqrt{1+n^2}$. and the pendulum swings clockwise. When the pendulum moves from A to D it loses the potential energy mwa, which is converted to kinetic energy. To supply energy as fast as possible to the pendulum it follows from Equation (5) that acceleration should be reversed when the pendulum reaches the point D. An observer in a coordinate frame fixed to the pivot then sees a gravitational field with strength w in the direction OC. The kinetic energy is continuous at the switch but the potential energy is discontinuous. The pendulum will swing towards the upright position if its kinetic energy is so large that it reaches the point E. The kinetic energy at F must thus be at least mbw. The condition for this is

$$a \geq b. \tag{10}$$

It follows from Figure 3 that $a = \sin\theta_0 - \cos\theta_0$ and $b = 1 - \sin\theta_0$. The condition (10) then becomes

$$2\sin\theta_0 \geq 1 + \cos\theta_0. \tag{11}$$

Introducing $n = \tan\theta_0$ and using equality in Equation (11) gives

$$2n = 1 + \sqrt{1+n^2}.$$

This equation has the solution $n = 4/3$. To have a single swing triple switch behavior the acceleration of the pivot must thus be at least $4g/3$. If $n = 4/3$ the pivot accelerates to the right until the pendulum reaches the horizontal. The pivot is then accelerated to the left until the desired energy is obtained. This happens when the pendulum is $30°$ from the vertical and the acceleration is then set to zero.

If n is greater than $4/3$ the acceleration of the pivot can be set to zero before the pendulum reaches the point E. Let θ^* be the angle of the pendulum when the acceleration of the pivot is set to zero. In a coordinate system fixed

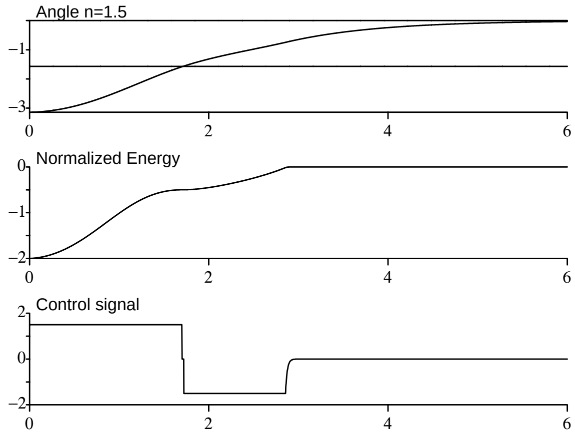

Figure 4 Simulation of the single-swing triple-switch control. The parameters are $n = 1.5$, $\omega_0 = 1$ and $k = 100$.

to the pivot the center of mass has then traveled the distance $l(2 - \sin\theta^*)$ in the horizontal direction. The force in the horizontal direction is mng. It follows from Equation (9) that the energy supplied to the pendulum is $mngl(2 - \sin\theta^*)$. Equating this with $2mgl$ gives $\sin\theta^* = 2(1 - 1/n)$.

EXAMPLE 2—SIMULATION OF SSTS BEHAVIOR
The single-swing triple-switch control is illustrated by the simulation shown in Figure 4. The swing-up is executed by simulating the normalized model with $\omega_0 = 1$. The control strategy is given by Equation (8) with parameters $n = 1.5$, and $k = 100$. The strategy is close to a pure switching strategy. The maximum control signal is applied initially. Energy increases but it has not reached the desired level when the pendulum is horizontal. To continue to supply energy to the pendulum the control signal is then reversed. The control signal is then set to zero when the desired energy is obtained. □

Multi-Swing Behavior

If the maximum acceleration is smaller than $4g/3$ it is necessary to swing the pendulum several times before it reaches the upright position. Let us first consider the conditions for bringing the pendulum up in two swings illustrated in Figure 5, which shows a coordinate system fixed to the pivot. An observer in this coordinate system sees a gravity field with strength $w = g\sqrt{1+n^2}$. The field has direction OB if the acceleration of the pivot is positive, and the direction OC when it is negative.

Assume that the pendulum starts at rest at A and that the pivot first accelerates to the right. The pendulum then swings from A to D. Acceleration is reversed when the pendulum reaches D and the pendulum then swings to the right around the line OD. The acceleration of the pivot is switched to the right when the pendulum reaches the horizontal position at E. To reach the upright position it is necessary that the pendulum can reach the point F without additional reversal of the acceleration. This is possible if its energy at E is sufficiently large to bring it up to point F. Consider the change of energy of the pendulum when it moves from rest at D to E. When it has moved to E it has lost the potential energy mwa, which has been transferred to kinetic energy. This kinetic energy must be sufficiently large to move the pendulum to F. This energy required for this is mwb, and we get the condition $a \geq b$. It follows from Figure 5

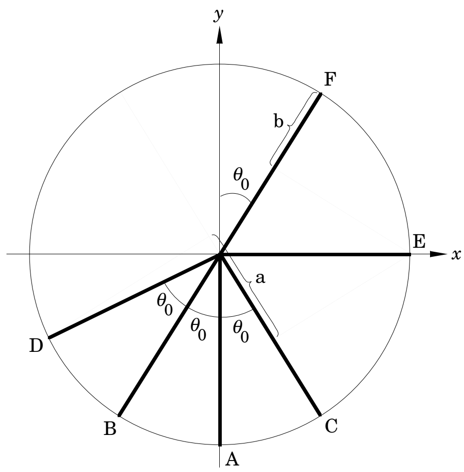

Figure 5 Figure used to derive the conditions for the double-swing quadruple-switch behavior. The origin of the coordinate system is called O.

that $a = \sin\theta_0 - \cos 3\theta_0$ and $b = 1 - \sin\theta_0$. Hence $\sin\theta_0 - \cos 3\theta_0 \geq 1 - \sin\theta_0$, which implies that

$$2\sin\theta_0 \geq 1 + \cos 3\theta_0. \tag{12}$$

The General Case

It is easy to extend the argument to cases where more swings are required. For example in a strategy with three swings the pendulum first swings $2\theta_0$ in one direction. Next time it swings $6\theta_0$ in the other direction, and the condition to reach the upright position becomes $2\sin\theta_0 \geq 1 + \cos 5\theta_0$. The corresponding equation for the case of k swings is

$$2\sin\theta_0 \geq 1 + \cos{(2k-1)}\theta_0. \tag{13}$$

Solving this equation numerically we obtain the the following relation between the acceleration of the pivot n and the number of swings k:

n	1.333	0.577	0.388	0.296	0.241	0.128
k	1	2	3	4	5	10

For small values of n the relation between n and k is approximately given by $n \approx \pi/(2k-1)$. Single swing behavior requires that $n > 4/3$, double swing behavior that $n > 0.577$. The number of swings required increases with decreasing n.

EXAMPLE 3—SIMULATION OF FIVE SWING BEHAVIOR

When $n = 0.25$ it follows from the table that five swings are required to bring the pendulum to the upright position. This is illustrated in the simulation shown in Figure 6. The process is simulated with the normalized equations with $\omega_0 = 1$. The control strategy given by Equation (8) is used with $n = 0.25$, and $k = 100$. □

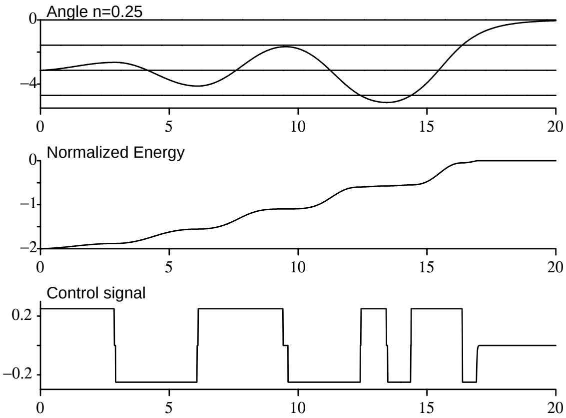

Figure 6 Simulation of energy control for the case $n = 0.25$, $\omega_0 = 1$ and $k = 100$. Five swings are required in this case.

Minimum Time Strategies

It follows from Pontryagins maximum principle that the minimum time strategies for swinging up the pendulum are of bang-bang type. It can be shown that the strategies have a nice interpretation as energy control. They will inject energy into the pendulum at maximum rate and then remove energy at maximum rate in such a way that the energy corresponds to the equilibrium energy when the upright position is reached. For small values of n the minimum time strategies give control signals that initially are identical with the strategies based on energy control. The final part of the control signals are, however, different because the strategies we have described will set the control signal to zero when the desired energy has been obtained. The strategies we have given can thus be described as strategies where there is no overshoot in the energy.

Consider for example the case $n > 2$, where a one-swing strategy can be used. To swing up the pendulum its energy must be increased with $2mgl$. This can be achieved by a single-swing double-switch strategy illustrated in Figure 2. The maximum acceleration is used until the pendulum has moved the angle $\arctan 2/n$. The energy can be increased further by continuing the acceleration, until the pendulum has reached the horizontal position. It follows from Equation (5) that the acceleration should then be reversed. By reversing the acceleration at a proper position the energy can then be reduced so that it reaches the desired value when the pendulum is horizontal. The energy is increased until it reaches a maximum value and it is then reduced at the maximum rate.

Let θ^* be the angle where the pendulum has its maximum energy. In a coordinate system fixed to the pivot the center of mass of the pendulum has traveled the distance $l(2 - \sin\theta^*)$ in the horizontal direction, when the energy is maximum. Since the horizontal force is mng it follows from Equation (9) that the energy supplied to the pendulum is $nmgl(2 - \sin\theta^*)$. To reduce the energy to zero when the pendulum is upright maximum deceleration is used for the distance $l\sin\theta^*$. This reduced the energy by $nmgl\sin\theta^*$. Since the energy required to swing up the pendulum is $2mgl$ we get

$$nmgl(2 - \sin\theta^*) - nmgl\sin\theta^* = 2mgl,$$

which implies that $\theta^* = \arcsin\left(1 - \frac{1}{n}\right)$. The maximum energy is

$$E_{max} = nmgl\sin\theta^* = (n-1)mgl. \tag{14}$$

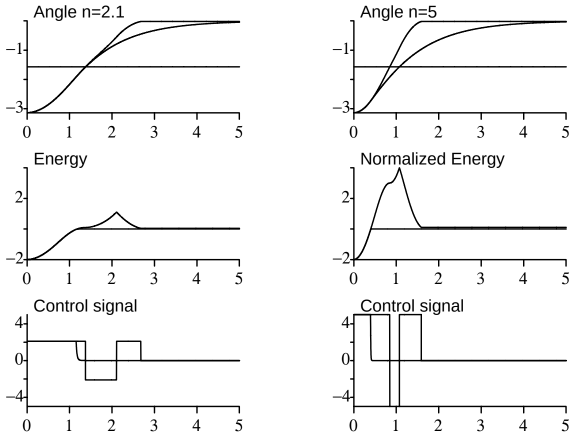

Figure 7 A comparison of energy control with minimum time control for $n = 2.1$ (left) and $n = 5$ (right).

For $n = 2$ the maximum energy is mgl. The "energy overshoot" is 50% for $n = 2$ and it increases rapidly with n. This explains why the minimum time strategies are sensitive for large n. Much energy is pumped into the system and dissipated as the pendulum approaches the upright position. Minor errors can give a substantial excess or deficit in energy. The energy control gives a much gentler control.

Figure 7 compares the minimum time strategies and the energy control strategies. The figure also shows that the difference in the time to reach the upright position increases with increasing n but the differences between $n = 2.1$ and $n = 5$ are not very large. It also shows that the minimum time strategy has an overshoot in the energy. With $n = 5$ it follows from Equation (14) that the maximum energy is $4mgl$ which is also visible in the simulation. The energy overshoot is more than 200%. □

Several different strategies are often combined to swing up the pendulum. A catching strategy is used when the pendulum is close to the upright position. The energy overshoot can actually be used as a robustness measure. A good practical approach is to use an energy control strategy with an energy excess of 10–20% and catch the pendulum when it is close to the upright position. Such a strategy is simple and quite robust to modeling errors. The idea has been used in many different laboratory experiments, see Iwashiro *et al.* (1996) and Eker and Åström (1996).

5. Generalizations

The energy control for a single pendulum is very simple. It leads to a first order system described by an integrator whose gain depends on the angle and its rate of change. The only difficulty is that the gain may vanish. This will only happen at isolated time instants because the time variation of the gain is generated by the motion of the pendulum. The ideas can be extended to control of more complicated configurations with rotating and multiple pendulums. In this section we will briefly discuss two generalizations.

General Dynamical System

To illustrate the ideas we consider a general mechanical system described by the equation

$$M(q,\dot{q})\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial U(q)}{\partial q} = T, \tag{15}$$

where q is a vector of generalized coordinates, $M(q,\dot{q})$ is the inertia matrix, $C(q,\dot{q})$ the damping matrix, $U(q)$ the potential energy and T the external control torques, see Marsden (1992). The total energy is

$$E = \frac{1}{2}\dot{q}' M(q,\dot{q})\dot{q} + U(q). \tag{16}$$

The time derivative of E is given by

$$\frac{dE}{dt} = \frac{1}{2}\dot{q}'(\dot{M}(q,\dot{q}) - 2C(q,\dot{q}))\dot{q} + T'\dot{q}. \tag{17}$$

In Spong and Vidyasagar (1989) it is shown that the matrix $\dot{M}(q,\dot{q}) - 2C(q,\dot{q})$ is skew symmetric. It thus follows that

$$\frac{dE}{dt} = T'\dot{q}.$$

The control torques depend on the control signal u and we thus have a problem of the type we have discussed previously. The problem is particularly simple if T is linear or affine in the control variable.

Two Pendulums

To illustrate the power of the method we consider two pendulums on a cart. The equations of motion for such a system are

$$\begin{aligned} \frac{dE_1}{dt} &= -m_1 u l_1 \dot{\theta}_1 \cos\theta_1 \\ \frac{dE_2}{dt} &= -m_2 u l_2 \dot{\theta}_2 \cos\theta_2. \end{aligned} \tag{18}$$

A control strategy that drives E_1 and E_2 to zero can be obtained from the Lyapunov function $V = (E_1^2 + E_2^2)/2$. The derivative of this function is

$$\frac{dV}{dt} = -Gu,$$

where

$$G = m_1 l_1 E_1 \dot{\theta}_1 \cos\theta_1 + m_2 l_2 E_2 \dot{\theta}_2 \cos\theta_2.$$

Provided that G is different from zero the control law

$$u = \mathrm{sat}_{ng} kG, \tag{19}$$

drives the Lyapunov function to zero. This implies that both pendulums will obtain their appropriate energies. The control law (19) will not work if the system is not controllable. This happens the pendulums are identical. It is then possible to have a motion with $\theta_1 + \theta_2 = 0$ which makes G equal to zero. A detailed discussion of the properties of the strategy is outside the scope of the paper.

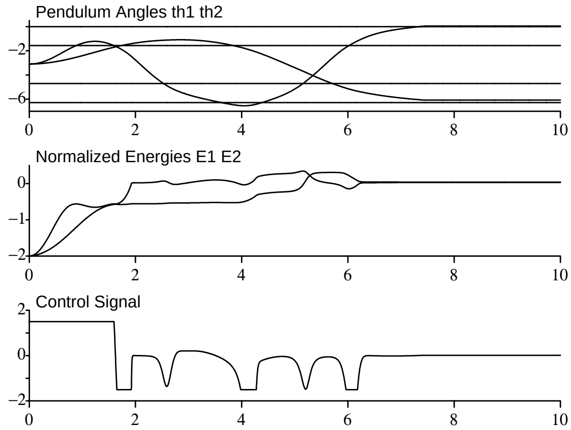

Figure 8 Simulation of the strategy for swinging up two pendulums on the same cart.

EXAMPLE 4—SWINGING UP TWO PENDULUMS ON A CART
Figure 8 illustrates swing-up of two pendulums with $\omega_{01} = 1$ and $\omega_{02} = 2$. The control strategy is given by Equation (19) with parameters $n = 1.5$ and $k = 20$. Notice that the control strategy brings the energies of both pendulums to their desired values. Also notice that the pendulums approach the upright position from different directions. The strategy swings up the pendulums much faster than the strategy proposed in Bortoff (1996). □

6. Conclusion

Energy control is a convenient way to swing up a pendulum. The behavior of such systems depend critically on one parameter, the maximum acceleration of the pivot. If the acceleration is sufficiently large, $u > 2g$, the pendulum can be brought to the upright position with one swing and two switches of the control signal. The control signal uses its maximum value until the desired energy is obtained and is then set to zero. If $4g/3 < u < 2g$ the pendulum can still be brought up with one swing, but the control signal now makes three switches. For lower accelerations the pendulum has to swing several times. The case of swinging up a simple pendulum has been treated in detail. The method has however also been applied to many other problems, for example swinging up of two pendulums on a cart and swinging up a double pendulum.

7. References

Abelson, C. F. (1996): "The effect of friction on stabilization of an inverted pendulum." Master thesis ISRN LUTFD2/TFRT--5563--SE. Department of Automatic Control, Lund Institute of Technology, Lund, Sweden.

Åström, K. J. (1998): "Hybrid control of inverted pendulums." In Yamamoto and Hara, Eds., *Intelligent and Hybrid Systems*. Springer, New York, N.Y.

Åström, K. J. (1999): "Hybrid control of inverted pendulums." In Yamamoto and Hara, Eds., *Learning, Control, and Hybrid Systems*, pp. 150–163. Springer.

Åström, K. J., K. Furuta, M. Iwashiro, and T. Hoshino (1995): "Energy based strategies for swinging up a double pendulum." In *IFAC World Congress*. Beijing, China.

Bortoff, S. A. (1996): "Robust swing-up control for a rotational double pendulum." In *IFAC'96, Preprints 13th World Congress of IFAC*, vol. F, pp. 413–419. San Francisco, California.

Chung, C. C. and J. Hauser (1995): "Nonlinear control of a swinging pendulum." *Automatica*, **31:6**, pp. 851–862.

Eker, J. and K. J. Åström (1996): "A nonlinear observer for the inverted pendulum." In *Proc. IEEE Conference on Control Applications*, pp. 332–337. Dearborn, Michigan.

Fradkov, A. L., P. Y. Guzenko, D. J. Hill, and A. Y. Pogromsky (1995): "Speed gradient control and passivity of nonlinear oscillators." In *Proc. IFAC Symposium on Control of Nonlinear Systems*, pp. 655–659. Lake Tahoe.

Fradkov, A. L., I. A. Makarov, A. S. Shiriaev, and O. P. Tomchina (1997): "Control of oscillations in hamiltonian systems." In *Proc. of the 4th European Control Conference*, pp. Paper TH–M G6. Brussels.

Fradkov, A. L. and A. Y. Pogromsky (1996): "Speed gradient control of chaotic continuous-time systems." *IEEE TCS*, **43**, pp. 907–913.

Furuta, K. and M. Yamakita (1991): "Swing up control of inverted pendulum." In *IEEE*, pp. 2193–2198. IECON'91.

Furuta, K., M. Yamakita, and S. Kobayashi (1992): "Swing-up control of inverted pendulum using pseudo-state feedback." *Journal of Systems and Control Engineering*, **206**, pp. 263–269.

Guckenheimer, J. (1995): "A robust hybrid stabilization strategy for equilibria." *IEEE Transactions on Automatic Control*, **40:2**, pp. 321–326.

Iwashiro, M., K. Furuta, and K. J. Åström (1996): "Energy based control of pendulum." In *Proc. IEEE Conference on Control Applications*, pp. 715–720. Dearborn, Michigan.

Krstić, M., I. Kanellakopoulos, and P. V. Kokotović (1994): "Passivity and parametric robustness of a new class of adaptive systems." *Automatica*, **30**, pp. 1703–1716.

Lin, Z., A. Saberi, M. Gutmann, and Y. A. Shamash (1996): "Linear controller for an inverted pendulum having restricted travel: A high-and-low approach." *Automatica*, **32:6**, pp. 933–937.

Lozano, R. and I. Fantoni (1998): "Passivity based control of the inverted pendulums." In Normand-Cyrot, Ed., *Perspectives in Control*. Springer, New York, London.

Maletinsky, W., M. F. Senning, and F. Wiederkehr (1981): "Observer based control of a double pendulum." In *IFAC World Congr.*, pp. 3383–3387.

Malmborg, J., B. Bernhardsson, and K. J. Åström (1996): "A stabilizing switching scheme for multi controller systems." In *IFAC'96, Preprints 13th World Congress of IFAC*, pp. 229–234. San Francisco, California.

Marsden, J. E. (1992): *Lectures on Mechanics.* Cambridge University Press, 1992, Cambridge, U.K.

Meier, H., Z. Farwig, and H. Unbehauen (1990): "Discrete computer control of a triple-inverted pendulum." *Opt. Cont. App. & Methods*, **11**, pp. 157–171.

Mori, S., H. Nishihara, and K. Furuta (1976): "Control of unstable mechanical systems: Control of pendulum." *Int. J. Control*, **23:5**, pp. 673–692.

Schaefer, J. F. and R. H. Cannon (1967): "On the control of unstable mechanical systems." In *Automat. Remote Contr. III, Proc. 3rd Int. Fed. Automat. Contr. (IFAC)*, vol. 1, 6C.1-6C.13.

Shinbrot, T., C. Grebogi, and J. Wisdom (1992): "Chaos in a double pendulum." *Amer. J. Phys.*, **60**, pp. 491–499.

Spong, M. W. (1995): "The swing up control problem for the acrobot." *IEEE Control Systems Magazine*, **15:1**, pp. 49–55.

Spong, M. W. and L. Praly (1995): "Energy based control of underactuated mechanical systems using switching and saturation." In Morse, Ed., *Preprints of the Block Island Workshop on Control Using Logic-Based Switching*, pp. 86–95. Rhode Island.

Spong, M. W. and M. Vidyasagar (1989): *Robot Dynamics and Control.* John Wiley & Sons.

Wei, Q., W. P. Dayawansa, and W. S. Levine (1995): "Nonlinear controller for an inverted pendulum having restricted travel." *Automatica*, **31:6**, pp. 841–850.

Wiklund, M., A. Kristenson, and K. J. Åström (1993): "A new strategy for swinging up an inverted pendulum." In *Preprints IFAC 12th World Congress*, vol. 9, pp. 151–154. Sydney, Australia.

Yamakita, M. and K. Furuta (1992): "VSS adaptive control based on nonlinar model for TITech pendulum." *IEEE*, pp. 1488–1493.

Yamakita, M., M. Iwashiro, Y. Sugahara, and K. Furuta (1995): "Robust swing up control of double pendulum." In *American Control Conference*, pp. 290–295.

Yamakita, M., K. Nonaka, and K. Furuta (1993): "Swing up control of a double pendulum." In *Proceedings of the American Control Conference*, pp. 2229–2233. San Francisco, California.

Yamakita, M., K. Nonaka, Y. Sugahara, and K. Furuta (1994): "Robust state transfer control of double pendulum." In *IFAC Symposium on Advances in Control Education, Boston*, pp. 223–226.